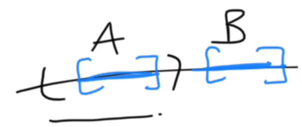
Connectedness

Def: Set $S \subset \mathbb{R}^n$ is connected if it cannot be written as $S = A \cup B$, where

(i) A, B are disjoint ie $A \cap B = \emptyset$
(ii) A, B both open (rel. to S)
(iii) $A \neq \emptyset$, $B \neq \emptyset$

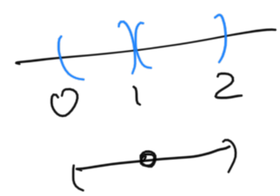


Non-ex

① $S = [0,1] \cup [2,3]$
not connected
$A = [0,1]$, $B = [2,3]$ $A \cap B = \emptyset$

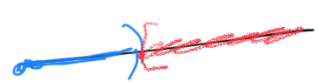
(i) $S = A \cup B$
(ii) $A = (-1/2, 3/2) \cap S$ So A rel open in S
$B = (3/2, 4) \cap S$

② $S = (0,1) \cup (1,2)$
$A = (0,1)$
$B = (1,2)$



→ ③ $S = \mathbb{Q}$ rationals
$S = (\mathbb{Q} \cap (-\infty, \pi)) \cup (\mathbb{Q} \cap (\pi, +\infty))$
($\mathbb{Q} \cap (-\infty, \pi)$ = A, $\mathbb{Q} \cap (\pi, +\infty)$ = B)

Ex: single pt
$S = \{0\} \subset \mathbb{R}$



Thm $[0,1]$ is connected

Pf: Suppose for contradiction
$[0,1] = A \cup B$, A, B satisfy (i), (ii), (iii)

a = largest element A
b = " " B

...?

Why do a, b exist?

A is open in $[0,1]$ $\Rightarrow$ B is closed in $[0,1]$

Last time: $[0,1]$ is compact, closed subset of cpt set,

so B is a closed subset of cpt set,

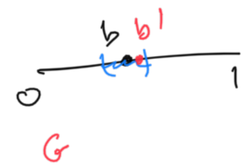
so B is compact.

$\Rightarrow$ B largest elt (otherwise $x_1 < x_2 < x_3 \cdots$; $\forall b' \in B$, $\exists x_i$ w/ $x_i > b$, $\in B$; seq has a limit pt b)

Similarly, A has largest elt.

WLOG, $a = 1$, $b \neq 1$

but B is open rel to $[0,1]$



so $\exists b' \in B$ w/ $b' > b$

contradiction, since b was largest elt of