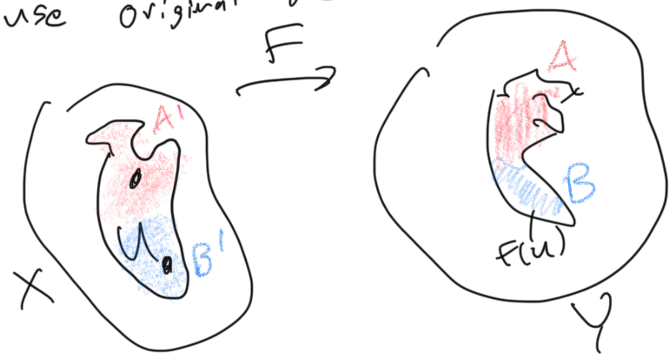
Connected sets and intermediate value theorem

Def: $X \subset \mathbb{R}^n$ is connected if there is no decomp
$X = A \cup B$ where
(i) $A \cap B = \emptyset$
(ii) A, B open in X
(iii) $A, B \neq \emptyset$

Thm If $f: X^{\subset \mathbb{R}^n} \to Y^{\subset \mathbb{R}^m}$ and f is continuous, $U \subset X$ is connected, then $f(U)$ is connected

We'll use original def. of continuity: ie $f^{-1}(V)$ is open if V open

Pf



Assume for contradiction that $f(U)$ not open
So $f(U) = A \cup B$ satisfying (i), (ii), (iii).

$\underbrace{U \cap f^{-1}(A)}_{A'}, \underbrace{U \cap f^{-1}(B)}_{B'}$

- By continuity of $f: U \to f(U)$, A', B' open (ii)
- Also, $U = A' \cup B'$
- $f^{-1}(A), f^{-1}(B)$ disjoint (since A, B are disjoint) $\Rightarrow A', B'$ are disjoint (i)
- B, A non-empty, both in $f(U)$ A', B' are non-empty (iii)

contradiction

$\Longrightarrow U$ is not connected, ...

Application

<u>Thm</u> (Intermediate Value) Let $f:[0,1] \to \mathbb{R}$ be a continuous function. If $f(0) = a$, $f(1) = b$ and c is between a, b, then there exists $x \in [0,1]$ w/ $f(x) = c$.

<u>Pf</u> • $[0,1]$ is connected

→ • By above $f([0,1])$ is also connected.

Suppose for contr. that c is missed.

WLOG $a < b$, $a < c < b$.

$V := f([0,1])$, $\quad V = \Big(\underbrace{V \cap (-\infty, c)}_{=A}\Big) \cup \Big(\underbrace{V \cap (c, +\infty)}_{=B}\Big)$

(i) $A \cap B = \emptyset$

(ii) A, B open in V

(iii) A, B non-empty since

$f(0) = a < c \quad f(1) \in B$

$f(0) \in A$

$\Longrightarrow V$ is not connected $\quad \boxed{\text{TL}}$