Basis for a topology

From last time:

Defn: X set, $\mathcal{T}$ = collection of subsets of X then $\mathcal{T}$ gives a topology if

(i) $X, \emptyset$ are in $\mathcal{T}$

(ii) Union of any collection of elts of $\mathcal{T}$ is in $\mathcal{T}$

(iii) intersection of any finite collection of elts of $\mathcal{T}$ is in $\mathcal{T}$

Def: $(X, \mathcal{T})$ topological space, say $C \subset X$ is closed if $X - C$ is open (i.e. is in $\mathcal{T}$)

(Finite unions of closed sets are closed
arbitrary intersections " ")

In $\mathbb{R}^n$, we had open balls $D^n(\underline{x}, r)$ special open sets

Def: Let $(X, \mathcal{T})$ top. space. Say $\mathcal{B} \subset \mathcal{T}$ is basis for $\mathcal{T}$ if

(i) $\forall x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$

(ii) If $x \in B_1, B_2$; $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ s.t. $B_3 \subset B_1 \cap B_2$ and $x \in B_3$

Def Suppose $\mathcal{B} \subset X$ satisfies (i), (ii) above, then the topology generated by $\mathcal{B}$ is the set $\mathcal{T}$ consisting of all $U \subset X$

U ...

w/ property that $\forall x \in U$, $\exists B \in \mathcal{B}$
w/ $x \in B$, $B \subset U$

<u>Claim</u>: $\mathcal{T}$ in above is in fact a topology

<u>Eg</u>: ① $X = \mathbb{R}^n$, $\mathcal{T}$ = open sets in $\mathbb{R}^n$
$\mathcal{B}$ = all balls $D^n(x, r)$ $\quad x \in \mathbb{R}^n$, $r \in \mathbb{R}_+$

② $X = \mathbb{R}^n$, $\mathcal{T}$ = open sets in $\mathbb{R}^n$
$\mathcal{B}$ = all balls $D^n(x, r)$ s.t all coordinates of x, r are <u>rational</u>.