Compactness via open covers

Def: X top space. An open cover of X is a collection $\mathcal{C}$ of open subsets of X s.t.

$$\longrightarrow \bigcup_{U \in \mathcal{C}} U = X.$$

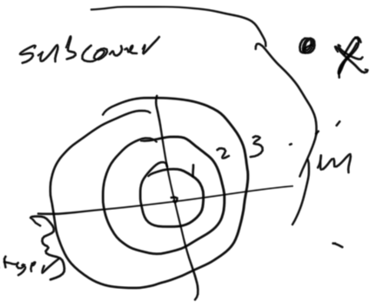
A subcover of $\mathcal{C}$ is a set $\mathcal{C}' \subset \mathcal{C}$ s.t. $\bigcup_{U \in \mathcal{C}'} U = X$.

Def A top. space X is compact if every open cover $\mathcal{C}$ has a finite subcover $\mathcal{C}'$

E.g. Any finite X (w/ any topology) is compact.

Each x_i is in some $U_i \in \mathcal{C}$

$X = U_1 \cup \dots \cup U_n$ finite subcover



Non. eg. $\mathbb{R}^n$ is not compact

$\mathcal{C} = \{ D^n(0,m) : m \text{ is a positive integer} \}$

$\mathcal{C}$ open cover of $\mathbb{R}^n$

If $\mathcal{C}' \subset \mathcal{C}$ finite subcover $\mathcal{C}'$ has k elts

$\mathcal{C}' = \{ D^n(0,m_1), \dots, D^n(0,m_k) \}$

$m = \max(m_1, \dots, m_k)$ positive integer

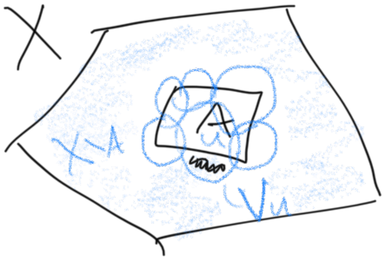
$(m, 0, \dots, 0) \in \mathbb{R}^n$ is not contained $\bigcup_{U \in \mathcal{C}'} U$

so $\mathcal{C}'$ does not cover all of $\mathbb{R}^n$.

Thm X compact top. space, $A \subset X$ closed in X,

(A is equipped w/ subspace.

then A is compact (A is [illegible] topology,



PF: Let $\mathcal{C}$ be an open cover of A

i.e. $A = \bigcup_{U \in \mathcal{C}} U$, $\underset{}{U} \overset{\subset A}{}$ open in A

Want: Get finite subcover of $\mathcal{C}$

Since U open in A, means

$U = A \cap V_U$ where $V_U \subset X$ open in X

Open Cover of X: $\{V_U\}_{U \in \mathcal{C}} \cup \{X \setminus A\}$

$\uparrow$ open

$\uparrow$ open in X, since A closed in X

By compactness of X:

Above cover has finite subcover

$V_{U_1} \cup V_{U_2} \cup \dots \cup V_{U_n} \cup \{X \setminus A\}$

$\longrightarrow X = V_{U_1} \cup V_{U_2} \cup \dots \cup V_{U_n} \cup \{X \setminus A\}$ $U_i \in \mathcal{C}$

But now $A \subset V_{U_1} \cup \dots \cup V_{U_n}$

$A = U_1 \cup \dots \cup U_n$

So $U_1, \dots, U_n$ finite subcover $\mathcal{C}$

So A compact.