Product spaces

<u>Def</u> X, Y sets, the product

$$X \times Y := \{ (x,y) : x \in X,\ y \in Y \}$$

<u>Eg</u>: ① $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

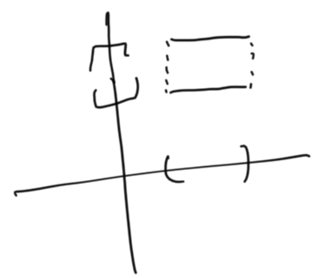
$\{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$

② X, Y finite sets

$|X \times Y| = |X| \cdot |Y|$

↑ # pts

③ $(1,2) \times [3,4]$



④ $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

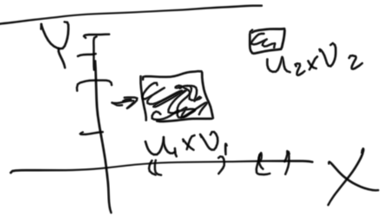
Q: X, Y top spaces, is there a natural topology on $X \times Y$ set?

<u>Def</u> $(X, T_X), (Y, T_Y)$ top spaces

(T_X: open sets)

The <u>product topology</u> on $X \times Y$ is defined by the <u>basis</u> $\mathcal{B}$ consisting of subsets of $X \times Y$ of form

$$U \times V, \quad U \in T_X, \quad V \in T_Y.$$



Note $(U_1 \times V_1) \cup (U_2 \times V_2)$ is not equal to $S \times T$ for any $S \subset X$, $T \subset Y$

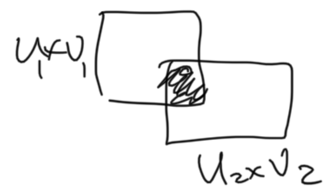
Need to check: $\mathcal{B}$ satisfies basis axioms

(i) every (x,y) is in some elt of $\mathcal{B}$
(take $U = X$, $V = Y$, $(x,y) \in U \times V$)

(ii) $(x,y) \in U_1 \times V_1,\ U_2 \times V_2$

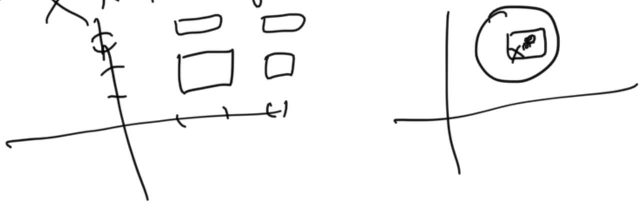
then $(x,y) \in \underbrace{(U_2 \cap U_1)}_{\text{open in } X} \times \underbrace{(V_1 \cap V_2)}_{\text{open in } Y}$

so in $\mathcal{B}$



Eg (1) $X = \mathbb{R}$, $Y = \mathbb{R}$ standard topology

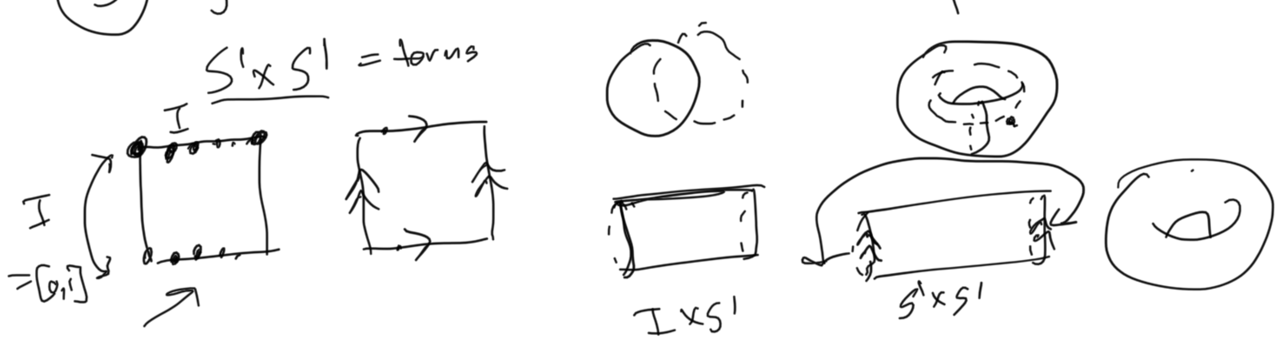
$\mathbb{R}^2 = X \times Y$ product topology is the usual topology on $\mathbb{R}^2$



(2) $[0,1] \times [0,1]$

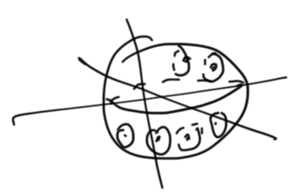
(3) S^1 = unit circle in $\mathbb{R}^2$

$S^1 \times S^1$ = torus



(4) S^2 = sphere in $\mathbb{R}^3$

$S^1 \times S^2$



(5) $S^1 \times S^1 \times S^1$ = 3-torus

$I \times I \times I$