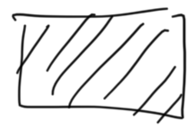
Properties of product spaces

Thm: X, Y top. spaces, connected, then $X \times Y$ is also connected.

Eg: (1) $I = [0,1]$ is connected $\Rightarrow I \times I$ is also conn'd



(2) S^1 is connected (e.g. since I is connected and S^1 is image of I under a contin. function).

$\Rightarrow S^1 \times S^1$ connected

(3) $\underbrace{S^1 \times S^1}\underbrace{\times S^1}$ 3-torus is connected

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Lemma: Let $X_1, X_2 \subset Z$ topological spaces X_1, X_2 connected, and $X_1 \cap X_2 \neq \emptyset$ Then $X_1 \cup X_2$ is connected.

(Holds for any $\bigcup_\alpha X_\alpha$, X_α is connected $\bigcap_\alpha X_\alpha \neq \emptyset$)

PF: To show $X_1 \cup X_2$ is connected, Suppose $X_1 \cup X_2 = U \cup V$, U, V open, disjoint, non-empty (for contradiction)

Let $z \in X_1 \cap X_2$.

One of U, V, WLOG U, contains z.

V non-empty, let $z' \in V$

Now, $V \subset X_1 \cup X_2$,

z' is in some X_i

Note $z \in X_i$

$$\underline{X_i} = (X_i \cap U) \cup (X_i \cap V) \ni z'$$

$z \in$

disjoint, open, ~~empty~~

$\Rightarrow X_i$ is not connected

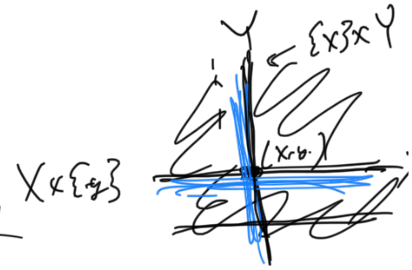
<u>Pf of Thm</u> X, Y connected

Choose $x \in X$, $y \in Y$

$A_y = \underline{X \times \{y\}} \subset X \times Y$

$B = \{x\} \times Y \subset X \times Y$

$(x,y) \in A \cap B$



A_y homeomorphic to X, so X A_y <u>is connected</u>

$\left(\begin{array}{l} X \times \{y\} \to X \\ (x,y) \mapsto x \\ \text{etc} \end{array} \right)$

B homeomorphic to Y, so B is <u>connected</u>

Lemma $\Rightarrow A_y \cup B$ is connected

$$X \times Y = \bigcup_{y \in Y} (A_y \cup B)$$

All $A_y \cup B$ have a point in common (any pt in B is in all of them)

So (stronger version) of Lemma

$$\Rightarrow X \times Y = \bigcup_{y \in Y} (A_y \cup B) \text{ is connected.}$$

<u>Thm</u> X, Y <u>compact</u> top spaces, then $X \times Y$ is also compact.

E.g. S^1 is compact $\Rightarrow S^1 \times S^1$ compact

$\Rightarrow S^1 \times S^1 \times S^1$ compact