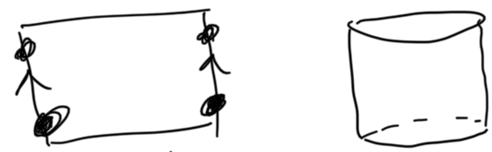
Quotient spaces

Motivation:



<u>Def</u> Let X be a set. An equivalence relation $\sim$ is a relation satisfying

(i) $x \sim x$ for any $x \in X$ (reflexive)

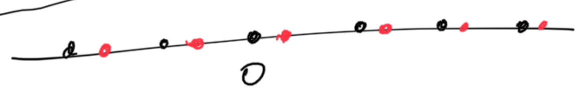
(ii) If $x \sim y$, then $y \sim x$ (symmetry)

(iii) If $\underline{x \sim y}$, $\underline{y \sim z}$, then $x \sim z$ (transitive)

<u>E.g</u> ① Normal notion of equality on X

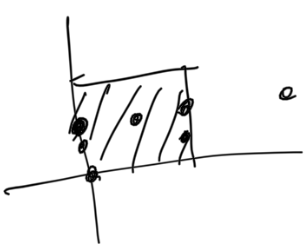
i.e. $x \sim y$ iff $x = y$

② $\underline{X = \mathbb{R}}$, $x \sim y$ iff $x - y$ is an integer



③ • X = unit square

$x \sim y$ iff $x = y$ or, x, y are on <u>opposite</u> sides and at same height.



• $X = \{(x,y) \in \mathbb{R}^2, 0 \le x \le 1, 0 \le y \le 1\}$

$\underline{(x_1, y_1)} \sim (x_2, y_2)$ iff $x_1 = x_2$ and $y_1 = y_2$

or $x_1 = 0$ or 1, and x_2 is the other

$y_1 = y_2$

④ X = All shapes in $\mathbb{R}^2$

$\sim$ congruence (Euclidean geometry)

⑤ X = all topological spaces

$\sim$ homeomorphic

transitive
$f: X \to Y$
$g: Y \to Z$

Def: X set, $\sim$ equivalence relation

The quotient of X by $\sim$ is

$$X/\sim \; := \{ [x] : x \in X \}$$

$$[x] := \{ y \in X : y \sim x \}$$ "equivalence class of x"

Eg ⑥ $X = \{1, 2, 3, 4\}$

$x \sim y$ iff $x - y$ is divisible by 2

$[1] = \{1, 3\}$ $\qquad X/\sim \; = \{ \{1,3\}, \{2,4\} \}$

$[2] = \{2, 4\}$

Natural map $p: X \to X/\sim$

$x \mapsto [x]$

Note: onto

Def: X topological space, $\sim$ equiv. reln on X

The quotient topology on $X/\sim$ is defined by

$U \subset X/\sim$ open iff $p^{-1}(U)$ is open in

$$\begin{array}{ccc} X & \xrightarrow{p} & X/\sim \\ \cup & & \cup \\ p^{-1}(U) & & U \end{array}$$

Ex: Prove that this defines a topology

Clear: p is continuous

E.g. (2) From above ~