Proof of class. of surfaces, part 3

Prop 1: Every connected compact surface is a connect sum of tori and projective planes (includes 0 summands $\rightsquigarrow$ sphere).

Proved last time

Want

Thm: Every connected compact surface is either
(i) a sphere
(ii) connect sum of n projective planes
(iii) " " n tori

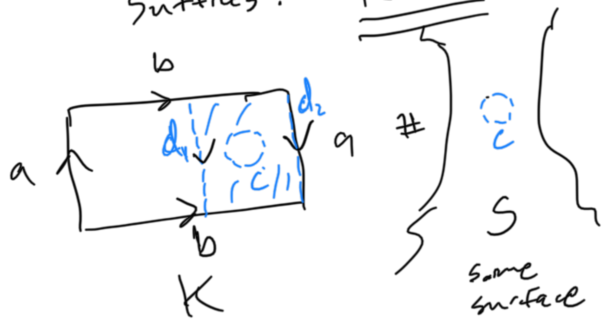
$\to$ **Lemma:** $\underline{\mathbb{P}^2 \# \mathbb{P}^2} \# \mathbb{P}^2 \cong \mathbb{P}^2 \# T$, T torus

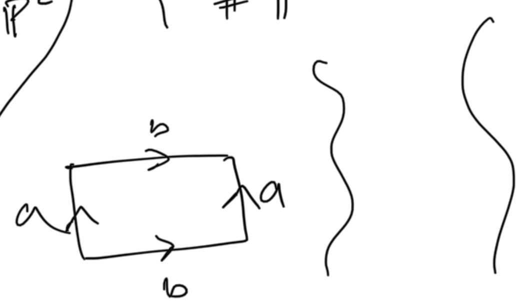
Note: Can't cancel $\mathbb{P}^2$: $\mathbb{P}^2 \# \mathbb{P}^2 \not\cong T$

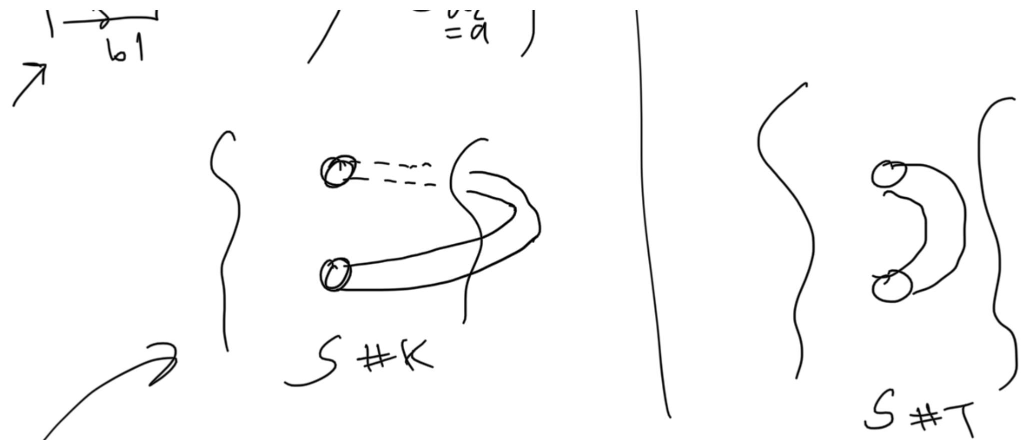
$\mathbb{P}^2 \# \mathbb{P}^2 \cong K$ Klein bottle

Pf of Lemma (Also in Kinsey)

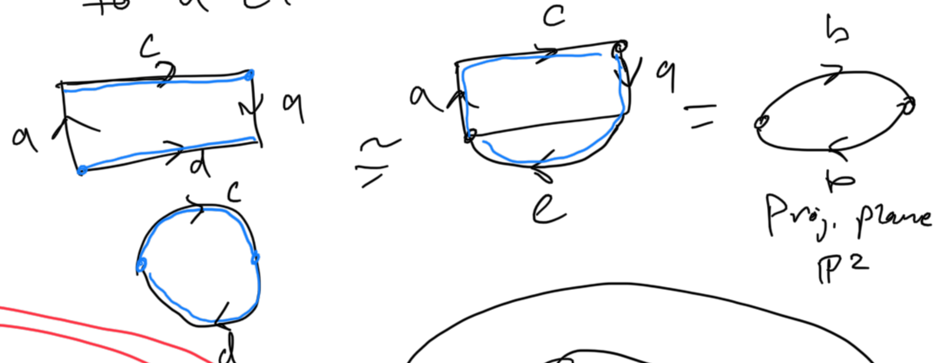
Suffices: $\underline{K \# \mathbb{P}^2} \cong T \# \mathbb{P}^2$ $\quad T \# \mathbb{P}^2$

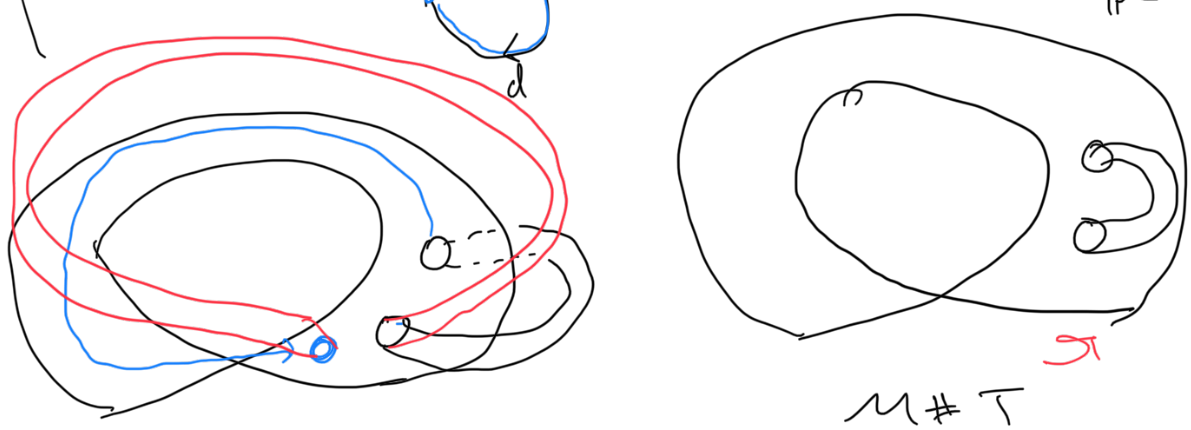






Note $\mathbb{P}^2$ = Möbius strip glued to a closed disk





$M \# K$ $\qquad$ $M \# T$

$$M \# K \cong M \# T$$

$\downarrow$ glue in disc $\qquad\qquad$ $\downarrow$ glue in disc

$$\Rightarrow \quad \underbrace{\mathbb{P}^2 \# K \cong \mathbb{P}^2 \# T}_{N \qquad\qquad N}$$

$\boxtimes$

Pf of Thm

By Prop 1, $M \cong \underbrace{\mathbb{P}^2 \# \cdots \# \mathbb{P}^2}_{j} \# \underbrace{T \# \cdots \#}_{k}$

If $j \geq 1$, replace ~~any~~ some $\mathbb{P}^2 \# T$ by $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$ (by Lemma)

This decreases num of T's, increases num of $\mathbb{P}^2$

Repeat $\rightsquigarrow$ get a connect sum of $\mathbb{P}^2$'s ✓

If $j = 0$, then already just a connect sum of tori ✓ ▨

E.g. $\mathbb{P}^2 \# \underbrace{\mathbb{P}^2 \# T} \# T$

$\cong \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \# \underbrace{\mathbb{P}^2 \# T}$

$= \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$