Graphs and trees

Def The Euler Characteristic $\chi(G)$ of a graph G is defined to be $v - e$, v = # vertices, e = # edges.

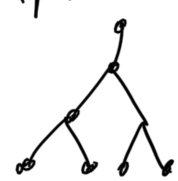
E.g. (1) Above has $e=4$, $v=4$,

$$\chi(G) = v - e = 4 - 4 = 0$$

(2)  $\chi(G) = v - e = 3 - 2 = 1$

cycle

Def A tree is a connected graph with no cycles.

E.g. (1) , (2) 

Non-examples: (1)  (2) 

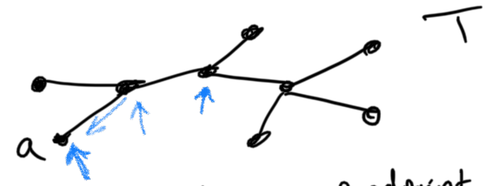
Thm If T a tree, then $\chi(T) = 1$.

Pf: Argue by induction on $e(T)$ ← # edges.

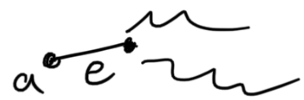
Base case: $e = 1$

$T =$ , $\chi(T) = 2 - 1 = 1$ ✓

<u>Inductive step</u>: Assume result for $n-1$ edge trees



Must have endpoint vertex a



Removing a and e gives a new tree T'

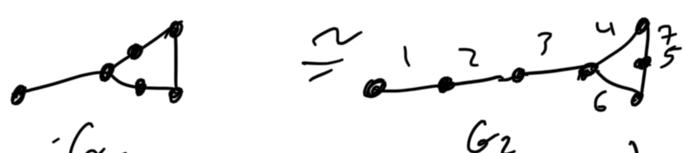
$e(T') = e(T) - 1$ $\quad \chi(T') = v(T') - e(T') = 1$

By induction,

$$\chi(T) = v(T) - e(T)$$
$$= \cancel{1} + v(T') - (\cancel{1} + e(T'))$$
$$= v(T') - e(T') = 1 \quad \checkmark$$

<u>Thm</u> If G_1, G_2 are graphs corresponding to homeomorphic spaces, then $\chi(G_1) = \chi(G_2)$.

<u>Eg</u>:



$\chi(G_1) = 6 - 6 = 0$, $\quad \chi(G_2) = 7 - 7 = 0$

<u>PF</u>: (1) <u>Branch vertex</u>

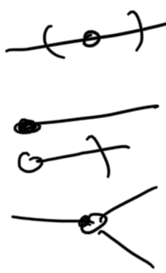
(3 or more edges)

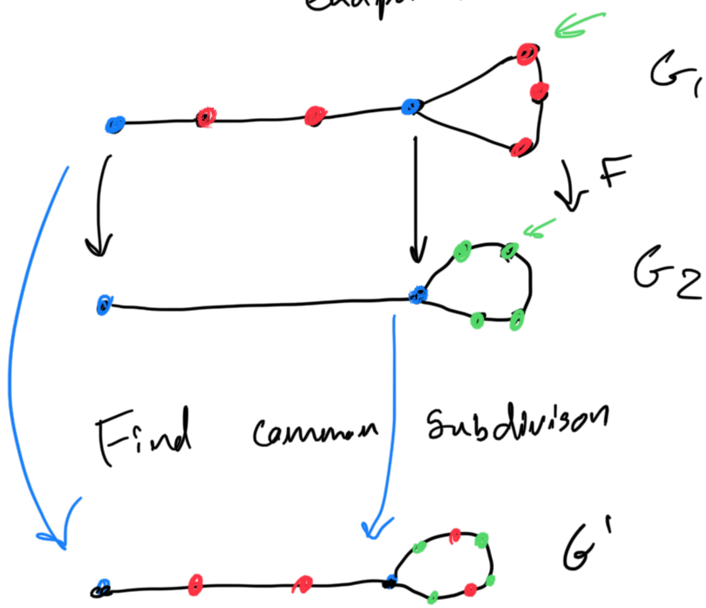
<u>End vertex</u>

(1 edge)

(2) Claim: If $f: G_1 \to G_2$ homeomorphism, f ... branch vertices to

then f takes branch vertices, and endpoints to endpoints.





Find common subdivison

This can be obtained from G_1 by adding in new vertices on existing edges. This increases edges by 1, vertices by 1, $\chi = v - e$ doesn't change under subdivision.

$$\chi(G_1) = \chi(G')$$
$$\chi(G_2) = \chi(G')$$
$$\Rightarrow \chi(G_1) = \chi(G_2)$$

$\blacksquare$