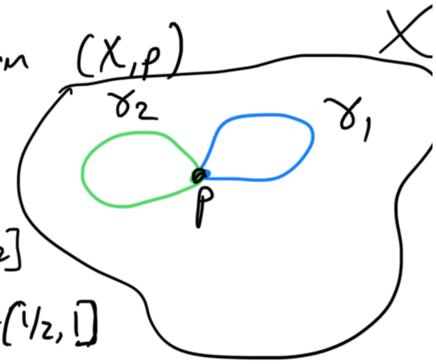
Fundamental group

Last time: defined $\pi_1(X,p)$ set of homotopy classes of based loops.

Additional structure: Composition

Def: Given γ_1, γ_2 based loops on (X,p)

define $\gamma_2 * \gamma_1 : [0,1] \to X$



$$(\gamma_2 * \gamma_1)(t) = \begin{cases} \gamma_1(2t), & t \in [0, 1/2] \\ \gamma_2(2t-1), & t \in (1/2, 1] \end{cases}$$

$$\gamma_1(2(1/2)) = \gamma_1(1) = p$$
$$\gamma_2(2(1/2)-1) = \gamma_2(0) = p$$
$$\gamma_2(2(1)-1) = \gamma_2(1) = p$$

Claim: $[\gamma_1] = [\gamma_1']$, $[\gamma_2] = [\gamma_2']$, then

$$[\gamma_2 * \gamma_1] = [\gamma_2' * \gamma_1']$$

i.e. composition well-defined on homotopy classes

So $*$ gives a binary operation on $\pi_1(X,p)$

$a \cdot b$

Def A group G is a set (G), together w/ a binary operation $\bullet : G \times G \to G$

Satisfying

(i) $\exists$ identity element $e \in G$, s.t.

$$x \cdot e = e \cdot x = x \quad \forall x \in G$$

(ii) associative $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

$\forall a, b, c \in G$

$\forall a,b,c \in G$

(iii) <u>inverses</u>: For each $g \in G$, $\exists h \in G$ s.t. $g \cdot h = h \cdot g = e$

<u>E.g.</u> ① $\mathbb{Z}$, w/ $\cdot = +$

② $\mathbb{R}$, w/ $\cdot = +$

③ $\mathbb{R} - \{0\}$, w/ $\cdot = \times$

④ Invertible matrices, w/ $\cdot =$ matrix mult.

⑤ $\mathbb{Z}/n$

<u>Claim</u>: $\pi_1(X, p)$, w/ $*$ is a <u>group</u>

C

For inverses
Want: $[\gamma_1] * [\gamma_2] = [e]$ — constant loop at p.

$$\pi_1(C, p) = \mathbb{Z}$$