Homework 4 : MATH 4180

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

1. Let $f : \mathbb{C} \to \mathbb{C}$ be given by

$$f(z) = \begin{cases} z^5/|z|^4 & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

   Write $f(x+iy) = u(x,y) + iv(x,y)$. Prove that the partials of u, v exist in a neighborhood of 0, and the Cauchy-Riemann equations hold at 0, but that f is not complex differentiable at 0. Why does this not contradict the theorem about Cauchy-Riemann equations?

2. Given $f : \mathbb{C} \to \mathbb{C}$ real-differentiable, define

$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right).$$

   Prove that $\partial/\partial\bar{z}$ satisfies the sum and product rules for differentiation. Show that the Cauchy-Riemann equations are equivalent to $\partial f/\partial \bar{z} = 0$.

3. Let $U = \mathbb{C} - \{z : \mathrm{Im}(z) = 0, \mathrm{Re}(z) \leq 0\}$ be the plane with non-positive real ray removed. Define $f : U \to \mathbb{C}$ in polar coordinates by

$$f(re^{i\theta}) = \log r + i\theta,$$

   where θ is chosen to lie in $(-\pi, \pi)$. Show this function is holomorphic and compute its derivative.

4. Compute the derivative of the local inverse of $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \sin z$ on a neighborhood of 0.

5. Is there an open set U in $\mathbb{C}$ such that $f : U \to \mathbb{C}$ given by $f(z) = \sin \bar{z}$ is holomorphic?