Homework 8: MATH 4180

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

1. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with $|f(\exp(\exp(z)))| \leq 1$ for all $z \in \mathbb{C}$. Prove that f is a constant function.

2. Let $U \subset \mathbb{C}$ be open, and $f : U \to \mathbb{C}$ a holomorphic function. Prove that there does not exist a point $z_0 \in U$ such that $|\exp(f(z_0))| < |\exp(f(z))|$ for all $z \in U - \{z_0\}$.

3. Find the maximum of $|\cos z|$ on $[0, 2\pi] \times [0, 2\pi]$.

4. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function, M a real number, and k a positive integer such that
$$|f(z)| \leq M|z|^k$$
for all $z \in \mathbb{C}$. Show that f is a polynomial of degree at most k.

5. Show that $u : \mathbb{C} \to \mathbb{R}$ given by $u(x, y) = e^{x^2-y^2} \sin(2xy)$ is harmonic. Find a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $\mathrm{Re}(f) = u$.