Homework 10: MATH 4180

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

1. Let $U \subset \mathbb{C}$ be open, $u_n : U \to \mathbb{R}$ harmonic functions, and $u : U \to \mathbb{R}$ some function such that the u_n converge to u uniformly. Prove that u is harmonic.

   You may use the following fact without proving it: if $U \subset \mathbb{C}$ open, and $h : U \to \mathbb{R}$ satisfies the Mean Value Property (i.e. for any disc $D(z, r)$ contained in U, the average value of h on the boundary of $D(z, r)$ is equal to $h(z)$), then h is harmonic.

2. Find the radius of convergence of the series $\sum_{n=1}^{\infty} n^5 z^n$. Does there exist a holomorphic function $f : B(0, 5) \to \mathbb{C}$ whose Taylor series is this power series?

3. Let $x > 1$ be a real number and let

   $$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

   Prove that there is an extension of ζ to a holomorphic function from $\{s \in \mathbb{C} : \text{Re}(s) > 1\}$ to $\mathbb{C}$ (that agrees with the above expression when s is a real number greater than 1).

4. Prove that the series

   $$\sum_{n=1}^{\infty} \frac{z^n}{1 + z^{2n}}$$

   converges in both the interior and exterior of the unit circle, and represents an analytic function in each region. Is there an entire function $f : \mathbb{C} \to \mathbb{C}$ that agrees with this series on both the interior and exterior of the unit circle?

   Bonus (5 points): Is there a connected open set $U \subset \mathbb{C}$ and a holomorphic function $f : U \to \mathbb{C}$ such that U intersects both the interior and exterior of the unit circle, and f agrees with the series above on $U \cap (\mathbb{C} - S^1)$?

5. Find an example of a power series $\sum_{n=0}^{\infty} a_n z^n$ with finite radius of convergence R such that the series converges uniformly on the closed ball $\overline{B(0, R)}$.