Homework 11: MATH 4180

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

1. Let n be a positive integer, and $p_1, \ldots, p_n \in D(0,1)$, with $p_i \neq 0$ for each i. Suppose $f : D(0,1) - \{p_1, \ldots, p_n\} \to \mathbb{C}$ is holomorphic. Let R be the radius of convergence of the Taylor series of f centered at 0. Show that if $R < 1$, then there exists some p_i that lies on $S^1(R)$, the circle of radius R centered at the origin, and f has a pole or essential singularity at p_i.

2. Show that there exists a pair of connected open sets $U_1, U_2 \subset \mathbb{C}$, two holomorphic functions $f_1 : U_1 \to \mathbb{C}$ and $f_2 : U_2 \to \mathbb{C}$, and two balls $D(p, \delta), D(p', \delta) \subset U_1 \cap U_2$ such that
   (i) $f_1(z) = f_2(z)$ for all $z \in D(p, \delta)$, and
   (ii) there exists $w \in D(p', \delta)$ such $f_1(w) \neq f_2(w)$.

3. Let $f, g : D(0,1) \to \mathbb{C}$ be holomorphic functions, such that g is not identically zero. Show that f/g is either holomorphic at 0, or has a pole there (in particular, it cannot have an essential singularity).

4. Let $f(z) : D(0,1)^* \to \mathbb{C}$ be given by
$$f(z) = \frac{\cos(e^z)}{z(z-1)\cos(z^2/100)}.$$
   Let $\sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent series for f on $D(0,1)^*$. Compute a_{-1}.

5. Compute the Laurent series for
$$f(z) = \frac{z+1}{z^5(z^2+1)}$$
   on the annulus $A_{0.1,0.9}$ centered at 0.