Lecture 1/27/22

Continue w/ complex pre-calculus

Last time: arithmetic over $\mathbb{C}$

Defn: For real y,

$$e^{iy} := \cos y + i \sin y.$$

How to define e^z, z any complex num?

Want: $e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$

$x, y \in \mathbb{R}$ real above

Defn: $e^{x+iy} := e^x(\underline{\cos y + i \sin y})$

$z \mapsto e^z = \exp(z)$

$\mathbb{C} \to \mathbb{C}$

Properties:

(i) Addition $\to$ Mult.

For any $w, z \in \mathbb{C}$

$$e^{z+w} = e^z e^w$$

Pf: $z = x_1 + iy_1$

$w = x_2 + iy_2$

$$e^{z+w} = e^{(x_1+x_2)+i(y_1+y_2)}$$

$$= \underline{e^{x_1+x_2}}(\cos(y_1+y_2) + i \sin(y_1+y_2))$$

$$= e^{x_1} e^{x_2} (\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2)$$

de Moivre

$$= e^z e^w \quad \checkmark$$

(ii) $e^{\pi i} = \cos\pi + i\sin\pi = -1$ Euler's formu

$e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1 = e^{0i}$

<u>Poll</u>: What is range of $\exp: \mathbb{C} \to \mathbb{C}$

i.e. $\exp(\mathbb{C})$?

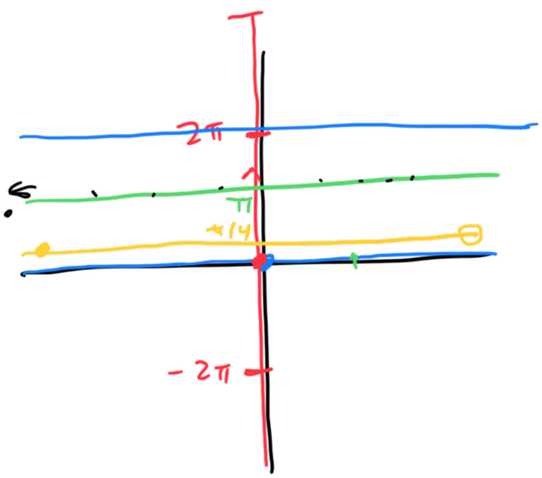
A) $\mathbb{C}$, B) $\mathbb{R}$, C) $\mathbb{R}-\{0\}$, D) $\mathbb{C}-\{0\}$

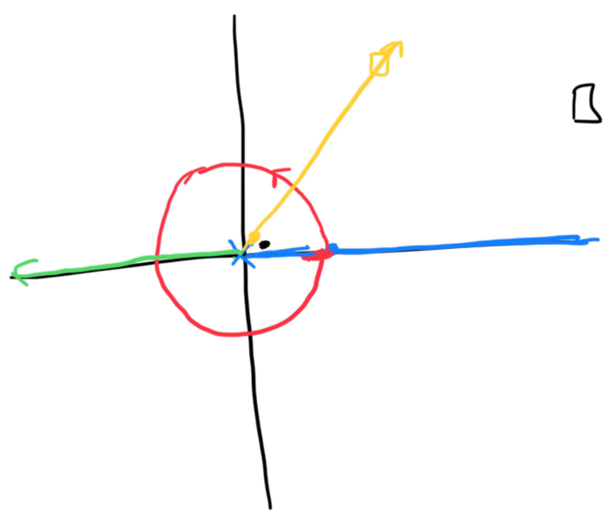
(iii) $e^z \neq 0$ for all $z \in \mathbb{C}$

(iv) $e^{z+2\pi i} = e^z$

Pf: Take $z = x+iy$

$$e^{x+i(2\pi+y)} = e^x(\cos(2\pi+y) + i\sin(2\pi+y))$$
$$= e^x(\cos y + i\sin y) = e^z$$





e^{iy} $z \longmapsto e^z$

$$e^{x+i\pi} = e^x(\cos\pi + i\sin\pi)$$
$$= -e^x$$

Change x

Trig functions w/ complex input

Have defined: $e^{iy} = \cos y + i\sin y$ $y \in \mathbb{R}$

$$e^{-iy} = \cos(-y) + i\sin(-y)$$
$$= \cos y - i\sin y$$

Solve for $\cos y$

$$e^{iy} + e^{-iy} = (\cos y + i\sin y) + (\cos y - i\sin y)$$
$$= 2\cos y$$

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

Use for any complex num.

Defn $z \in \mathbb{C}$, $\cos z := \dfrac{e^{iz} + e^{-iz}}{2}$

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i}$$

E.g. ① $\cos(i) = \dfrac{e^{i^2} + e^{-i^2}}{2} = \dfrac{e^{-1} + e}{2}$

$$1 < \frac{e^{-1} + e}{2} < 2$$

② $\cos(10i) = \dfrac{e^{i(10i)} + e^{-i(10i)}}{2} = \dfrac{e^{-10} + e^{10}}{2}$

Big real number

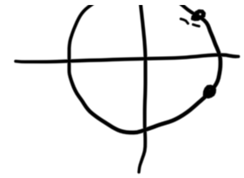
$\cos : \mathbb{C} \to \mathbb{C}$ is <u>unbounded</u>

Poll Q: Is $\cos(z)$ real for all $z \in \mathbb{C}$?

A) Yes, B) No

$e^{i(a+bi)}, e^{-i(a+bi)}$

$$\cos(a+bi) = \frac{e \qquad + e}{2}$$



$$= \frac{e^{-b}e^{ia} + e^{b}e^{-ia}}{2}$$

$b = 1000$ $\quad \approx \dfrac{e^{b}e^{-ia}}{2}$ not real

<u>Trig identities:</u>

(i) $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$

<u>PF</u> $\left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2$

$$= \frac{e^{2iz} - 2 + e^{-2iz}}{-4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4}$$

$$= \frac{4}{4} = 1 \quad \checkmark$$

(ii) sum identities, double angle, ...

Define $\tan = \sin/\cos$

<u>Logarithms</u>

Want: log is inverse of exp

$\exp(\log z) = z, \quad \log \exp(z) = z$

$r \in \mathbb{R}, \ \theta \in \mathbb{R}$

<u>Guess:</u> $z = re^{i\theta} = r(\cos\theta + i\sin\theta) \leftarrow$

$\log z = \log re^{i\theta} \underset{\uparrow}{=} \log r + \log e^{i\theta} \leftarrow$

Assume, since $e^{z+w} = e^{z}e^{w}$

$= \log r + \underline{\underline{i\theta}} \leftarrow$

<u>Issues</u> (1) If $r = 0$, $\log r$ is not defined

(2) $re^{i\theta} = re^{i(\theta + 2\pi)}$

log log

$\log r + i\theta$ $\log r + i(\theta + 2\pi)$

θ not well-defined

One-solution to defining $\log(z)$, $z \neq 0$

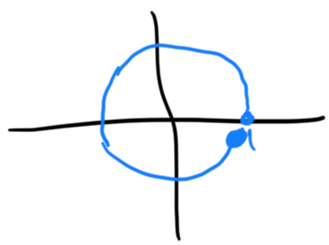
Write $z = re^{i\theta}$, $\theta \in [0, 2\pi)$

expression is unique

Then $\log z := \log r + i\theta$

Note: With this defn, $\log : \mathbb{C} - \{0\} \to \mathbb{C}$

discontinuous at positive real ray



$$\log\left(\cos(2\pi - 0.1) + i\sin(2\pi - 0.1)\right)$$

$$= i(2\pi - 0.1)$$

$1 = 1 \cdot e^0$

$\log 1 = 0 + 0 = 0$

$c \in \mathbb{R}$

$\overline{c\,z} = c\,\overline{z}$