Lecture 2/1/22

Review: $z = re^{i\theta}$

$$\log z = \log re^{i\theta} = \log r + \log e^{i\theta}$$
$$= \log r + i\theta$$

not well-defined
can add multiples of 2π

"multi-valued function"

Exponents

Q: Take $z, w \in \mathbb{C}$. What is w^z?

Know for: • $z \in \mathbb{Z}$ • e^z

• $(-1)^{1/2} = i, -i,$

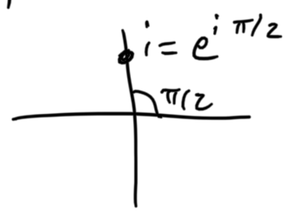
Unknown. $(3+i)^{1/2}$, $(3+i)^{\sqrt{2}}$, $(3+i)^{i+5}$

Idea: $w^z = e^q$ "change base $w \rightsquigarrow e$"

$$\log w^z = \log e^q$$
$$z \log w = q$$

Def: $w^z := e^{z \log w}$

Note: $\log w$ is multi-valued, and w^z is in general as well



E.g. $i^i = e^{i \log i}$

$$\log i = 0 + i \cdot (\pi/2 + n \cdot 2\pi), \quad n \in \mathbb{Z}$$

$$i^i = e^{i \log i} = e^{i \cdot i (\pi/2 + n \cdot 2\pi)} = e^{-\pi/2 - n \cdot 2\pi}$$

$n \in \mathbb{Z}$

∴ Infinitely many possible values of i^i

infinitely

Prop: If $z \in \mathbb{Z}$, then w^z is well-defined.

Pf: $w^z = e^{z \log w}$, $\quad w = re^{i\theta}$

$\log w = \log r + i(\theta + n \cdot 2\pi)$

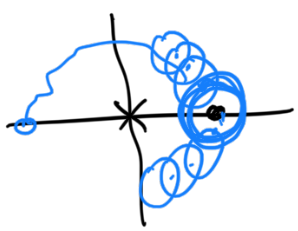
$= e^{z(\log r + i(\theta + n \cdot 2\pi))}$

$= e^{z \log r} e^{z i \theta} e^{\underline{i n \cdot 2\pi z}}$, $\quad nz$ is an integer (since n, z are) $\Rightarrow$

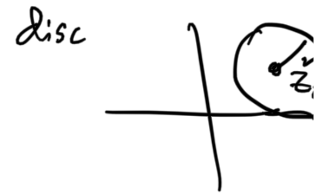
$= e^{z \log r} e^{z i \theta} \cdot 1$

does <u>not</u> depend n, so w^z well-defined

Domain of log



Continuity and Limits



$z_0 \in \mathbb{C}$, $r > 0$

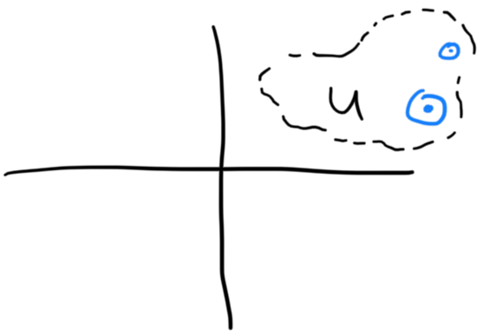
Def: Disc $D(z_0, r) := \{ z \in \mathbb{C} : |z - z_0| < r \}$.

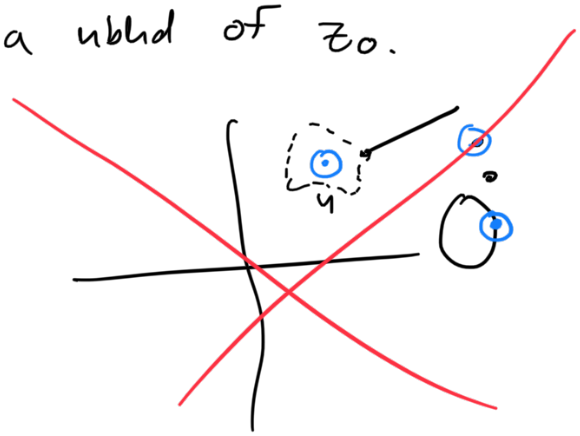
Punctured disc $D^*(z_0, r) := \{ z \in \mathbb{C} - \{z_0\} : |z - z_0| < r \}$

$= D(z_0, r) - \{z_0\}$

Def: A <u>neighborhood</u> N of $z_0 \in \mathbb{C}$ is a subset of $\mathbb{C}$ containing $D(z_0, r)$ for some $r > 0$.

Def: A set $U \subset \mathbb{C}$ is <u>open</u> if $\forall z_0 \in U$, U is a nbhd of z_0.





Limits:

Def. Take $U \subset \mathbb{C}$ open set, $f: U \to \mathbb{C}$, $z_0 \in \mathbb{C}$.
Then $\lim_{z \to z_0} f(z) = w$ if $\forall \varepsilon > 0$, $\underline{\exists \delta > 0}$ s.t. $z \in D^*(z_0, \delta)$, then $|f(z) - w| < \varepsilon$

Note: No condition on $f(z_0)$.

Continuity:

Def: $U \subset \mathbb{C}$ open, $f: U \to \mathbb{C}$. Then f is <u>continuous</u> if $\lim_{z \to z_0} f(z) = f(z_0)$ $\forall z_0 \in U$.

Functions are continuous:

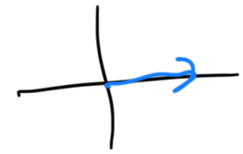
E.g. Many familiar functions

polynomials, exp

Non-e.g. $\log : \mathbb{C}^* \to \mathbb{C}$

$\mathbb{C}^* = \mathbb{C} - \{0\}$

$z = re^{i\theta} \mapsto \log r + i\theta$

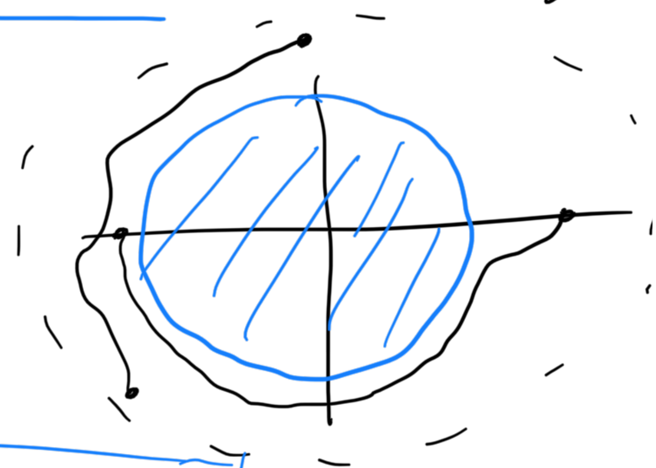


$\theta \in [0, 2\pi)$

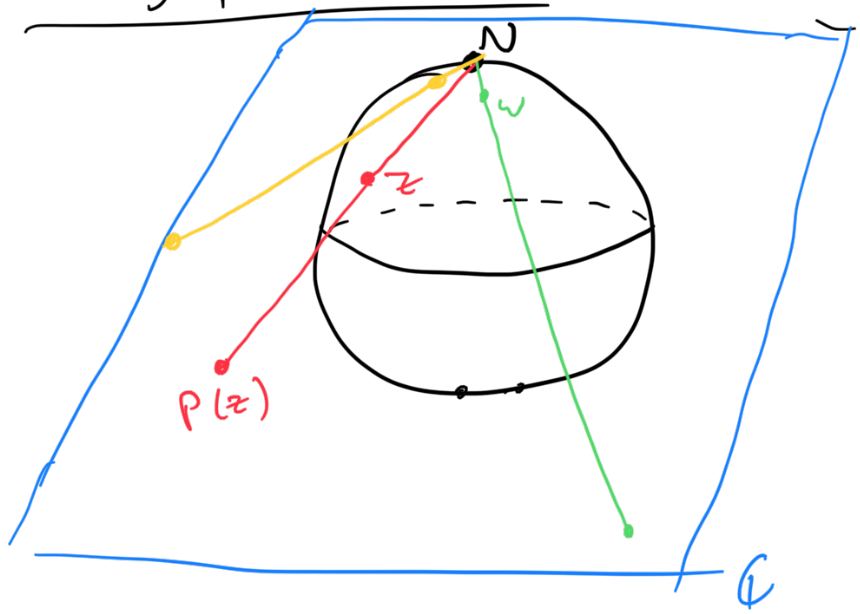
Not continuous: jump at positive real ray

Riemann Sphere

$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$



Stereographic Projection

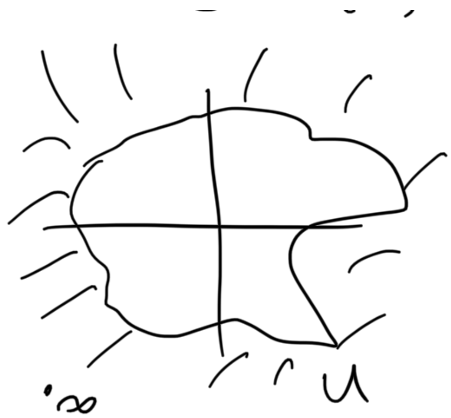


$p : \text{Sphere} \to \hat{\mathbb{C}}$

$p(N) = \infty$

What is a nbhd of ∞ in $\hat{\mathbb{C}}$?

Def: A set $U \subset \hat{\mathbb{C}}$ w/ $\infty \in U$ is a neighborhood of ∞ if it contains $\mathbb{C} - D(z_0, R)$ for some R.



Arithmetic on $\hat{\mathbb{C}}$

For any $z \in \mathbb{C}$

1. $z + \infty = \infty$
2. $z \cdot \infty = \infty, \quad z \neq 0$
3. $\infty \cdot \infty = \infty$
4. $\dfrac{z}{\infty} = 0$

But can't evaluate: $0 \cdot \infty$, $\underline{\infty/\infty}$, $\infty - \infty$