Lecture 2/3

- Directed Reading Program: <u>deadline 2/4</u>
  Cornell DRP
- Next week, class is in Malott 207

Last time: Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Warning: $\infty + \infty$ <u>indeterminate</u>

$$1000 + (-1000) = 0$$

$|z|$ is big   small abs. value

Poll: Evaluate $\lim_{z \to \infty} \frac{z^3+17z}{z^2+11}$ in $\hat{\mathbb{C}}$

A) 1
B) 17
C) ∞
D) does not exist in $\hat{\mathbb{C}}$

$|z|$ big

$z^3+17z \sim z^3$

$z^2+11 \sim z^2$

$\frac{z^3}{z^2} \sim z$

$$= \lim_{z\to\infty} \frac{z + 17/z}{1 + 11/z^2} = \lim_{z \to \infty} z = \infty$$

Complex derivatives

Def: $f: U \to \mathbb{C}$, U open subset of $\mathbb{C}$,
say f is <u>complex differentiable</u> at $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \text{ exists (in } \mathbb{C}\text{)},$$

complex

limit

it's value, when it exists, is the complex derivative $f'(z_0), \left(\frac{d}{dz}f\right)(z_0)$ of f at z_0

Note: Recall $\lim\limits_{z \to z_0}$ defn wasn't sensitive to value at z_0

Real differentiable $\mathbb{R} \to \mathbb{R}$ $\iff$ well-approximated by line $\iff$ fn well-approx by $x \mapsto ax + b$, $a, b \in \mathbb{R}$

Complex differentiable $\mathbb{C} \to \mathbb{C}$ $\iff$ well-approx by $z \mapsto az + b$, $a \in \mathbb{C}$, $b \in \mathbb{C}$

Def: $f : U \to \mathbb{C}$, $U \subset \mathbb{C}$ open, say f is holomorphic if complex diff'ble at z_0 $\forall z_0 \in U$.

Eg: $f : \mathbb{C} \to \mathbb{C}$

$f(z) = az + b$, $a \in \mathbb{C}$, $b \in \mathbb{C}$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{az + \cancel{b} - (az_0 + \cancel{b})}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{a\cancel{(z - z_0)}}{\cancel{z - z_0}} = \lim_{z \to z_0} a = a$$

so f is holomorphic

$f'(z_0) = \frac{d}{dz} f(z_0) = a \quad \forall z_0 \in \mathbb{C}$

Prop: If $f : U \to \mathbb{C}$ is complex diff'ble at z_0, then f is continuous at z_0.

Pf: want: $\lim\limits_{z \to z_0} (f(z) - f(z_0)) = 0$

know: $\lim\limits_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = a \in \mathbb{C}$

$$= \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \cdot (z-z_0)$$

$$= \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \cdot \lim_{z \to z_0} (z-z_0)$$

$$= a \cdot 0 = 0$$ ▨

<u>Linearity</u>: $f, g : U \to \mathbb{C}$ holomorphic
$a, b \in \mathbb{C}$

then $af + bg : U \to \mathbb{C}$ is also holomorphic,

and $(af+bg)'(z_0) = a f'(z_0) + b g'(z_0)$. $\forall z_0 \in$

<u>PF</u>: $$\lim_{z \to z_0} \frac{(af(z)+bg(z)) - [af(z_0)+bg(z_0)]}{z-z_0}$$

$$= \lim_{z \to z_0} \left[\frac{a(f(z)-f(z_0))}{z-z_0} + \frac{b(g(z)-g(z_0))}{z-z_0} \right]$$

$$= a \lim \frac{f(z)-f(z_0)}{z-z_0} + b \lim \frac{g(z)-g(z_0)}{z-z_0}$$

$$= a f'(z_0) + b g'(z_0).$$

<u>Product Rule</u>: $f, g : U \to \mathbb{C}$ holomorphic

then $fg : U \to \mathbb{C}$ holomorphic

and $(fg)'(z_0) = f'(z_0) g(z_0) + f(z_0) g'(z_0)$
$\forall z_0 \in$

<u>PF</u>: $$\lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z-z_0}$$

$$= \lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z)}{z - z_0} + \frac{f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \left[g(z) \cdot \frac{f(z) - f(z_0)}{z - z_0} \right] + f(z_0) \lim \frac{g(z) - g(z_0)}{z - z_0}$$

$$= \left(\lim_{z \to z_0} g(z) \right) \cdot f'(z_0) + f(z_0)\, g'(z_0)$$

$$= g(z_0) f'(z_0) + f(z_0) g'(z_0) \qquad \blacksquare$$

by diff'ble $\Rightarrow$ cts

Polynomials:

Prop: If $f : \mathbb{C} \to \mathbb{C}$, $f(z) = a_n z^n + \dots + a_1 z + a_0$, $a_i \in \mathbb{C}$, then f is holomorphic,

$$f'(z) = a_n \cdot n \cdot z^{n-1} + \dots + a_1$$

PF sketch: $(z^2)' = (z \cdot z)' = (z)' \cdot z + z \cdot (z)'$

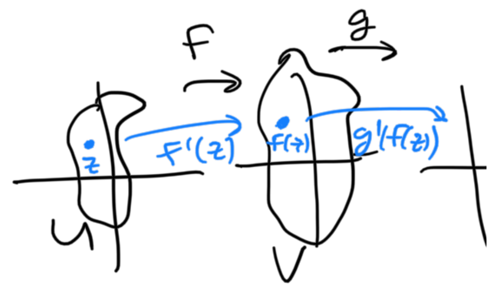
$$= 1 \cdot z + z \cdot 1 = 2z$$

By induction, prove $(z^n)' = n z^{n-1}$ $\square$

Chain Rule $f : U \to \mathbb{C}$

$g : V \to \mathbb{C}$

$f(U) \subset V$ (subset)



f, g holomorphic

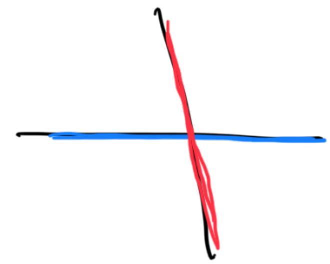
Then $g \circ f$ (ie $z \mapsto g(f(z))$) is holomorphic

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z).$$

Notation: $\subset$ → subset
$\subsetneq$ strict subset

Non-eg: $f: \mathbb{C} \to \mathbb{C}$, $f(z) = \mathrm{Re}(z)$
is not complex differentiable at 0.

Pf: $\lim\limits_{\substack{x \to 0 \\ x \in \mathbb{R}}} \frac{f(x) - f(0)}{x - 0}$ $\qquad \frac{x-0}{x-0} = 1$



$= \lim\limits_{\substack{x \to 0 \\ x \in \mathbb{R}}} \frac{x - 0}{x - 0} = 1$

real limit

$\frac{-y-0}{y} = -1$

$\lim\limits_{\substack{y \to 0 \\ y \in i\mathbb{R}}} \frac{f(y) - f(0)}{y - 0} = \lim\limits_{\substack{y \to 0 \\ y \in i\mathbb{R}}} \frac{0 - 0}{y} = \lim\limits_{\substack{y \to 0 \\ y \in i\mathbb{R}}} 0 = 0$

$\{z : z = iy, y \in \mathbb{R}\}$

So $\lim\limits_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ does not exist!

complex limit

So f not complex diff'ble at 0.

Warning: This f is real-differentiable as a fn $\mathbb{R}^2 \to \mathbb{R}^2$

Poll: Is $f(z) = \bar{z}$, $f: \mathbb{C} \to \mathbb{C}$, holomorphic?

A) Yes, B) No

Sol: If yes, then define $g(z) := \frac{1}{2}(z + \bar{z})$
$= \mathrm{Re}(z)$

would also be holomorphic

(by linearity, and fact that $z \mapsto z$ is hol

contradicting previous non-e.g.