Math 4180 (Complex Analysis) Course Notes, Spring 2022

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Contents

1 Basic theory of complex numbers

1.1 Complex numbers

Defining property of i: $i^2 = -1$.

Definition 1.1. Arithmetic with complex numbers:

1. Addition: $(a_1 + b_1 i) + (a_2 + b_2 i) := (a_1 + a_2) + (b_1 + b_2)i$
2. Multiplication: $(a_1 + b_1 i) \cdot (a_2 + b_2 i) := (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i$.

Form a *field*: associative, commutative, distributive, has identity, inverses.

Geometric interpretation of addition. Addition of vectors in $\mathbb{R}^2$.

Polar coordinates. Write $a + bi = r(\cos\theta + i\sin\theta)$.
Here $r = |z|$, and we call θ the *argument* $\arg(z)$ of z.
(Note that angle $\theta = \arg(z)$ is only well-defined up to multiples of 2π.)

Multiplication formula:

$$r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) = (r_1 r_2)(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Can think of multiplication by z as a map $\phi_z : \mathbb{R}^2 \to \mathbb{R}^2$, $w \mapsto zw$.
Q: What linear operators on $\mathbb{R}^2$ arise in this way? A: Those that are compositions of rotation and scaling.

De Moivre's formula: If $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Geometric interpretation of $z \mapsto z^2$: circle gets wrapped around itself twice.

Complex conjugation: $\overline{a + bi} := a - bi$.

1.2 Exponential and trigonometric functions

Exponential function. Recall that for real x:

$$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \cdots$$

We can attempt to use the formula for an imaginary number iy:

$$\begin{aligned} e^{iy} &= 1 + (iy) + (iy)^2/2! + (iy)^3/3! + (iy)^4/4! + \cdots \\ &= (1 - y^2/2! + y^4/4! - \cdots) + i(y - y^3/3! + y^5/5! - \cdots) \\ &= \cos y + i \sin y. \end{aligned}$$

So we take this to be the definition for imaginary arguments, i.e. for $\theta \in \mathbb{R}$, we have

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Then, we would like $e^{x+iy} = e^x e^{iy}$, and combined with the above, this defines the exponential function for all complex arguments.

Properties:

- Addition `-->` multiplication: $e^{z+w} = e^z e^w$.
- Euler's formula: $e^{\pi i} = -1$.
- Periodicity: $e^{z+2\pi i} = e^z$.
- $e^z \neq 0$ for all $z \in \mathbb{C}$.

Trig functions. From the above, for real y, we have $e^{iy} = \cos y + i \sin y$, and then using the symmetry properties of $\sin, \cos$, we get $e^{-iy} = \cos y - i \sin y$. Solving for $\sin, \cos$, we get

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \quad \cos y = \frac{e^{iy} + e^{-iy}}{2}.$$

We can then use these same formulas to define $\sin z, \cos z$ for any *complex* z, since we've already defined e^z.

Example: $\cos(10i) = \frac{e^{-10}+e^{10}}{2}$. Note this is a big real number; $\cos, \sin$ are *unbounded* on $\mathbb{C}$.

Trig identities (e.g. $\sin^2 z + \cos^2 z = 1$) also hold in complex world, but must be checked.

Geometric picture of $z \mapsto e^z$: takes a horizontal strip to the punctured plane.

1.3 Logarithm

Want to define log as the inverse of exp. How can we guess the fomula? Should satisfy $\log(wz) = \log w + \log z$.

To define $\log z$, begin by writing $z = re^{i\theta}$. Then

$$\log re^{i\theta} = \log r + \log e^{i\theta} = \log r + i\theta.$$

Take this to be the definition. Can check that $e^{\log z} = z$, as desired. (Note that this doesn't work for $z = 0$, since $\log 0$ is undefined).

Major Issue: The angle θ is not actually well-defined, e.g. $e^{i\theta} = e^{i(\theta+2\pi)}$. Can take $\theta \in [0, 2\pi)$, but then log is not continuous. This is called a *branch* of the logarithm.

1.4 Complex exponents

How to define w^z for $w, z \in \mathbb{C}$. Since we know how to define e^z, we just need to "change the base": $w^z = e^q$ so $z \log w = q$, so

$$w^z := e^{z \log w}.$$

Since $\log w$ is multi-valued, so is this. But if z is an integer it is well-defined, since the different values of $\log w$ won't change the value of $e^{z \log w}$.

1.5 Topology of $\mathbb{C}$

Definition 1.2. Given $z_0 \in \mathbb{C}$ and $r > 0$, the *(open) disc* $D(z_0, r)$ is the set $\{z \in \mathbb{C} : |z - z_0| < r\}$.

The *punctured disc* $D^*(z_0, r)$ is $D(z_0, r) - \{z_0\}$.

Definition 1.3. A *neighborhood* N of $z_0 \in \mathbb{C}$ is a subset of $\mathbb{C}$ containing $D(z_0, r)$ for some $r > 0$.

Definition 1.4. A set $U \subset \mathbb{C}$ is *open* if $\forall z_0 \in U$, U is a neighborhood of z_0.

Definition 1.5 (Limits)**.** Given $U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}$, and $z_0 \in \mathbb{C}$, we write

$$\lim_{z \to z_0} f(z) = w$$

if $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $z \in D^*(z_0, \delta)$, then $|f(z) - w| < \epsilon$.

Definition 1.6 (Continuity)**.** Given $U \subset \mathbb{C}$ open, and $f : U \to \mathbb{C}$, we say f is *continuous* if $\forall z_0 \in U$,

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Riemann sphere. As a set, the Riemann sphere $\widehat{\mathbb{C}}$ is just $\mathbb{C} \cup \{\infty\}$. Its topology is determined by the following.

Definition 1.7. A set $U \subset \mathbb{C}$ with $\infty \in U$ is a *neighborhood of* ∞ if it contains $\mathbb{C} - D(z_0, R)$ for some $R > 0$.

Many arithmetic operations extend to $\widehat{\mathbb{C}}$:

- $z + \infty = \infty$ for any $z \in \mathbb{C}$
- $z \cdot \infty = \infty$ for any $z \in \mathbb{C} - \{0\}$
- $\infty \cdot \infty = \infty$
- $z/\infty = 0$ for any $z \in \mathbb{C}$.

On the other hand, the following expressions are indeterminate:
$\infty + \infty, \infty/\infty, 0 \cdot \infty$.

2 Complex differentiation

2.1 Definition and basic properties

Definition 2.1. Let $U \subset \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$. We say f is *complex differentiable* at $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, and if it does the value above is the *complex derivative* at z_0 (denoted f' or $\frac{d}{dz} f$).

Means: f locally well-approximated by complex scaling.

Example 2.2. Given $a, b \in \mathbb{C}$, the function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = az+b$ is complex differentiable at any $z_0 \in \mathbb{C}$, and its derivative is equal to a (at any z_0).

Proposition 2.3. *If $f : U \to \mathbb{C}$ is differentiable at z_0, then f is continuous at z_0.*

Proof. We have

$$\begin{aligned} \lim_{z \to z_0} (f(z) - f(z_0)) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot (z - z_0) \\ &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0) \\ &= 0. \end{aligned}$$

■

Definition 2.4. Let $U \subset \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$. We say f is *holomorphic* if it is complex differentiable at all $z_0 \in U$.

Proposition 2.5 (Linearity)**.** *If $a, b \in \mathbb{C}$, $f, g : U \to \mathbb{C}$ are holomorphic, then $af + bg$ is holomorphic and*

$$(af + bg)' = af' + bg'.$$

Proposition 2.6 (Product Rule)**.** *If $f, g : U \to \mathbb{C}$ are holomorphic, then so is fg, and*

$$(fg)' = f'g + fg'.$$

Proof. We have

$$\begin{aligned}\lim_{z\to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} &= \lim_{z\to z_0} \frac{f(z)g(z) - f(z_0)g(z)}{z - z_0} + \lim_{z\to z_0} \frac{f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0} \\ &= \lim_{z\to z_0} g(z) \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} + f(z_0) \lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0} \\ &= g(z_0)f'(z_0) + f(z_0)g'(z_0).\end{aligned}$$

■

Proposition 2.7 (Polynomial derivatives)**.** *Any polynomial with complex coefficients gives a holomorphic function, and*

$$(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)' = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots + a_1.$$

Proof. Use linearity, product rule, and derivative of $az + b$. ■

Proposition 2.8 (Chain rule)**.** *Let $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be holomorphic functions, with $f(U) \subset V$. Then $g \circ f : U \to \mathbb{C}$ (i.e. the function $z \mapsto g(f(z))$) is holomorphic and*

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z).$$

Example 2.9. The function $f : \mathbb{C} \to \mathbb{C}$, $f(z) := \operatorname{Re} z$ is *not* complex differentiable at 0.

Proof. If z approaches 0 along the real-axis, the difference quotient approaches 1, while along the imaginary axis, it approaches 0. Hence the complex limit of difference quotients cannot exist. ■

Real differentiability.

Definition 2.10. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. We say that f is *real differentiable* at $x_0 \in \mathbb{R}^n$ if there exists a $m \times n$ matrix A such that $\forall \epsilon > 0$, $\exists \delta$ such that

$$\|(f(x) - f(x_0)) - A(x - x_0)\| \leq \epsilon \|x - x_0\|$$

whenever $\|x - x_0\| < \delta$. In this case, $Df|_{x_0} := A$ is the derivative of f at x_0 (note that it is easy to see that such an A is unique if it exists).

Derivatives of curves. We can apply the above to a plane curve, i.e. a real differentiable map $\gamma : \mathbb{R}^1 \to \mathbb{R}^2$. We can write $\gamma(t) = (x(t), y(t))^T$, and then the derivative is $\gamma'(t) = (x'(t), y'(t))^T$. In complex notation, we write this as $x'(t) + iy'(t)$. Geometrically, this is the *tangent vector* at $\gamma(t)$ (provided that $\gamma'(t) \neq 0$).

Proposition 2.11. *Let $f : U \to \mathbb{C}$ be a holomorphic, and $\gamma : \mathbb{R} \to \mathbb{C}$ real differentiable, with $\gamma(\mathbb{R}) \subset U$. Then $f \circ \gamma$ is real differentiable and*

$$(f \circ \gamma)'(t) = f'(\gamma(t)) \cdot \gamma'(t).$$

2.2 Conformality

Definition 2.12. Given $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{C}$ differentiable with $\gamma_1(0) = \gamma_2(0) = z_0$ and $\gamma_1'(0), \gamma_2'(0) \neq 0$, we define the *angle* between γ_1, γ_2 at z_0 to be

$$\arg \gamma_1'(0) - \arg \gamma_2'(0) \mod 2\pi$$

Definition 2.13. Let $U \subset \mathbb{C}$ open, and $f : U \to \mathbb{C}$ real differentiable. We say if f is *conformal* at $z_0 \in U$ if $Df|_{z_0}$ is non-singular (i.e. $\det(Df|_{z_0}) \neq 0$), and for any pair $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{C}$ with $\gamma_1(0) = \gamma_2(0) = z_0$, $\gamma_1'(0), \gamma_2'(0) \neq 0$, we have:

$$\begin{aligned}&\text{angle between } \gamma_1, \gamma_2 \text{ at } z_0\\ =&\text{angle between } f \circ \gamma_1, f \circ \gamma_2 \text{ at } f(z_0).\end{aligned}$$

Proposition 2.14. *If $f : U \to \mathbb{C}$ is complex differentiable at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0.*

Proof. For any γ_1, γ_2, we have, by the chain rule Proposition 2.11:

$$\begin{aligned}\text{angle between } f \circ \gamma_1, f \circ \gamma_2 \text{ at } f(z_0) &= \arg(f \circ \gamma_1)'(0) - \arg(f \circ \gamma_2)'(0)\\ &= \arg((f'(z_0)\gamma_1'(0)) - \arg((f'(z_0)\gamma_2'(0))\\ &= (\arg f'(z_0) + \arg \gamma_1'(0)) - (\arg f'(z_0) + \arg \gamma_2'(0))\\ &= \arg \gamma_1'(0) - \arg \gamma_2'(0)\\ &= \text{angle between } \gamma_1, \gamma_2 \text{ at } z_0.\end{aligned}$$

■

Example 2.15. Consider the map $f(z) = z^2$. It is holomorphic, and $f'(z) = 2z$. The proposition above says it is conformal, except possibly at 0. One can check that at 0, angles are doubled, so it is in fact not conformal there.

2.3 Cauchy-Riemann equations

Theorem 2.16 (Cauchy-Riemann equations)**.** *Let $U \subset \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$. Write $f(x+iy) = u(x,y)+iv(x,y)$. Then f is complex differentiable at z_0 if and only if (i) f is real-differentiable at z_0, and (ii) f satisfies the* Cauchy-Riemann equations*:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

where the partials are evaluated at z_0. When f is complex differentiable at z_0, we have $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

Proof. Being complex differentiable at z_0 means that the function is locally well-approximated by complex scaling. This is equivalent to being (i) real differentiable, and (ii) the 2×2 Jacobian derivative matrix at z_0 corresponding to complex scaling. A 2×2 matrix A corresponds to scaling by some complex number iff the diagonal entries are equal and the off-diagonal entries are negatives of one another. So in terms of the Jacobian matrix, this corresponds to the Cauchy-Riemann equations above. ■

To check that a function is complex differentiable using the Cauchy-Riemann equations, we first need to establish that it is real differentiable. For this the following fact from multivariable real analysis is often useful:

Fact 2.17. Take $U \subset \mathbb{R}$ open, and $f : U \to \mathbb{R}$. If all partials $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$ exist and are continuous everywhere on U, then f is real differentiable everywhere on U.

Remark 2.18. The existence of the partials alone is *not* enough to guarantee real differentiability. One sees this by considering the function defined by $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. The function behaves quite differently on different rays through the origin, and so is not even continuous at 0.

Example 2.19. Take $f(z) = z^2$. We write $f(x+iy) = (x+iy)^2 = (x^2-y^2) + i(2xy)$, so $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. Then we compute

$$\partial u/\partial x = 2x, \quad \partial u/\partial y = -2y$$
$$\partial v/\partial x = 2y, \quad \partial v/\partial y = 2x.$$

The partials are continuous everywhere and satisfy the Cauchy-Riemann equations everywhere, so by Fact 2.17 and Theorem 2.16, f is complex differentiable everywhere.

Proposition 2.20. *The function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = e^z$ is complex differentiable everywhere (i.e. it is holomorphic), and $f'(z) = e^z$.*

Proof. We will apply the Theorem 2.16 (Cauchy-Riemann equations). We compute the partials

$$\partial u/\partial x = e^x \cos y, \quad \partial u/\partial y = -e^x \sin y$$
$$\partial v/\partial x = e^x \sin y, \quad \partial v/\partial y = e^x \cos y.$$

These exist, and are continuous, so by Fact 2.17 (applied to each component function), we get that f is real differentiable, establishing (i) of the Cauchy-Riemann condition, and (ii) can be seen from the above partials computation. Hence f is complex differentiable everywhere, and

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + e^x \sin y = e^z.$$

■

We can then use the above to compute complex derivatives of $\sin, \cos$ (the formulas are the same as those found in calculus).

Cauchy-Riemann equations in polar form. Writing $z = re^{i\theta}$ and $f(z) = u(r,\theta) + iv(r,\theta)$, we can express the Cauchy-Riemann equations in "Polar-Cartesian" form (polar coordinates are being used for the domain, while cartesian is being used for range):

Theorem 2.21. *Let $U \subset \mathbb{C}^* = \mathbb{C} - \{0\}$ be open and $f : \mathbb{C}^* \to \mathbb{C}$ be given by $f(z) = u(r,\theta) + iv(r,\theta)$. Then f is complex differentiable at z_0 if and only if (i) f is real-differentiable at z_0, and*

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}, \quad \textit{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r},$$

where the partials are evaluated at z_0.

When f is complex differentiable at z_0 we have $f'(z_0) = \frac{1}{iz_0}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$.

Proof. These can be deduced from the cartesian Cauchy-Riemann equations using the Chain Rule. In class, we gave a more geometric argument using infinitesimals. ■

2.4 Derivatives of inverse functions

Differentiating the logarithm. Let $U = \mathbb{C} - \{z : \mathrm{Im}(z) = 0, \mathrm{Re}(z) \geq 0\}$ be the plane with non-negative real ray removed. We can define the "standard branch" of log on this domain by $\log z = \log re^{i\theta} = \log r + i\theta$, where θ is chosen to lie in $(0, 2\pi)$.

We will apply Theorem 2.21 to this branch on U. Real differentiability can be checked by writing r, θ in terms of x, y (note that there is an issue at the origin, but $0 \notin U$). And then it easy to compute the partials and check they satisfy the equations. So this $\log z$ is holomorphic and

$$\frac{d}{dz}\log z = \frac{1}{iz}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) = \frac{1}{iz}(0 + i \cdot 1) = \frac{1}{z}.$$

Theorem 2.22 (Holomorphic Inverse Function)**.** *Let $U \subset \mathbb{C}$ be open, and $f : U \to \mathbb{C}$ holomorphic. Let $z_0 \in U$ with $f'(z_0) \neq 0$. Then there exist neighborhoods V of z_0 and W of $f(z_0)$ such that $f(V) = W$, $f|_V : V \to W$ is a bijection, and $f^{-1} : W \to V$ is a holomorphic.*

Let f be as above. Then by the Chain Rule applied to $f(g(w)) = w$ we get that $g'(w) = 1/f'(g(w))$, i.e.

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}.$$

Applying with $\exp, \log$ (on suitable domains, and with a suitable branch of log), we get

$$\frac{d}{dw}\log w = \frac{1}{\exp(\log w)} = \frac{1}{w}.$$

3 Complex Integration

3.1 Definitions and basic properties

Definition 3.1 (Integral of complex function over interval)**.** Let $g : [a, b] \to \mathbb{C}$ continuous. Write $g(t) = u(t) + iv(t)$. Then

$$\int_a^b g(t)dx := \int_a^b u(t) + i\int_a^b v(t).$$

Definition 3.2 (Contour integral)**.** Let $U \subset \mathbb{C}$ open, and let $f : U \to \mathbb{C}$ be continuous. Let $\gamma : [a, b] \to U$ be a C^1 curve (continuously differentiable). Then the *contour integral* of f along γ is defined by

$$\int_\gamma f dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Remark 3.3. The definition above naturally extends to piecewise C^1 curves. Our theorems about contour integrals will typically be stated for C^1 curves, but most can be generalized to the case of piecewise C^1 curves. We will sometimes use these more general results in applications.

The contour integral can also be defined as a limit of Riemann sums

$$\lim_{\Delta z \to 0} \sum_i f(z_i)(z_{i+1} - z_i),$$

where z_i are points on the curve, and the maximum distance Δz between consecutive points goes to 0.

Definition 3.4. The C^1 curves $\gamma_1 : [a_1, b_1] \to \mathbb{C}$, $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ are *reparametrizations* of one another if there exists a continuously differentiable $\alpha : [a_1, b_1] \to [a_2, b_2]$ with $\alpha(a_1) = a_2$, $\alpha(b_1) = b_2$, $\alpha'(t) > 0$ for all t, and

$$\gamma_1 = \gamma_2 \circ \alpha.$$

Proposition 3.5 (Independence of reparametrization)**.** *If γ_1, γ_2 as in the previous definition are reparametrizations of one another, and $f : U \to \mathbb{C}$ is a continuous function (whose domain U contains images of γ_1, γ_2), then*

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

Proof. Use the Chain Rule, and the change of variables rule for integration. ∎

The above proposition means that the integral is well-defined on "oriented geometric curves"; it is not sensitive to the particular parametrization.

Here is the first (and one of the only) contour integrals that we will do "by hand".

Example 3.6. Consider the unit circle with counter-clockwise orientation. We wish to compute the contour integral of $f(z) = 1/z$ along this. We choose the natural unit speed parametrization $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(\theta) = e^{i\theta}$, and use the definition of contour integral:

$$\int_\gamma \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} i e^{i\theta} d\theta = 2\pi i.$$

Theorem 3.7 (Fundamental theorem of calculus for contour integrals)**.** *Let $\gamma : [0, 1] \to \mathbb{C}$ be a C^1 curve. Let $F : U \to \mathbb{C}$ be a holomorphic function, where $U \subset \mathbb{C}$ is open and contains the image of γ. Assume that F' is continuous. Then*

$$\int_\gamma F'(z) dz = F(\gamma(1)) - F(\gamma(0)).$$

Proof. Write $F(\gamma(t)) = u(t) + iv(t)$. We have

$$\begin{aligned}
\int_\gamma F'(z) dz &= \int_\gamma F'(\gamma(t))\gamma'(t) dt = \int_0^1 \frac{d}{dt} F(\gamma(t)) dt \\
&= \int_0^1 (u'(t) + iv'(t))\, dt = \int_0^1 u'(t) dt + i \int_0^1 v'(t) dt \\
&= (u(1) - u(0)) + i\,(v(1) - v(0)) \\
&= F(\gamma(1)) - F(\gamma(0)),
\end{aligned}$$

where in the second-to-last step, we have used the standard one-variable Fundamental Theorem of Calculus separately on u, v. ∎

Examples: integrals of polynomials and z^{-n} for $n \geq 2$.

Corollary 3.8. *If Γ is a closed loop (start and end point are the same), then $\int_\Gamma F'(z)dz = 0$.*

Corollary 3.9. *The contour integral of $F'(z)$ is the same over any two oriented paths γ_1, γ_2 going from z to z'.*

A converse to the above Corollary also holds:

Theorem 3.10 (Antiderivative from path independence)**.** *Let $f : U \to \mathbb{C}$ be continuous, with $U \subset \mathbb{C}$ open. Suppose contour integrals of f are* path-independent*, meaning that for any $z, z' \in U$ and any γ_1, γ_2 paths in U from z to z', we have $\int_{\gamma_1} f dz = \int_{\gamma_2} f dz$. Then f has a global antiderivative on U, i.e there exists $F : U \to \mathbb{C}$ holomorphic with*

$$F'(z) = f(z), \ \forall z \in U.$$

Proof. Pick a point $z_0 \in U$. Define

$$F(z) := \int_\gamma f(\zeta) d\zeta,$$

where γ is some path from z_0 to z. By hypothesis, this contour integral is independent of the particular γ chosen.

Now we check that the derivative at z is $f(z)$. By continuity of f, for any $\epsilon > 0$, we can find δ such that if $w \in D(z, \delta)$, then $|f(z) - f(w)| < \epsilon$. Choose a straight segment α from z to w. Then

$$\left| f(z) - \frac{F(w) - F(z)}{w - z} \right| = \left| \frac{f(z)(w - z) - \int_\alpha f(\zeta) d\zeta}{w - z} \right| = \left| \frac{\int_\alpha \left(f(z) - f(\zeta)\right) d\zeta}{w - z} \right|$$
$$\leq \frac{\epsilon \cdot \text{length}(\alpha)}{|w - z|} = \epsilon,$$

where in the second to last step we have used the Lemma below. This implies that $F'(z) = f(z)$.

(If U has multiple connected components, we should deal separately with each one, choosing a basepoint in each.) ■

Lemma 3.11 (Triangle inequality for integrals)**.** *If $f : U \to \mathbb{C}$ is continuous, then*

$$\left| \int_\gamma f(z) dz \right| \leq \text{length}(\gamma) \cdot \max_{z \in \gamma} |f(z)|.$$

Proof. Use Riemann sum expression of integral, and triangle inequality. ■

3.2 Line integrals

Recall the definition of a line integral of a real-valued differential form $P(x,y)dx + Q(x,y)dy$ over a C^1 curve $\gamma : [0,1] \to \mathbb{R}^2$:

$$\int_\gamma P(x,y)dx + Q(x,y)dy := \int_0^1 P(x(t),y(t))x'(t)dt + \int_0^1 Q(x(t),y(t))y'(t)dt,$$

where $\gamma(t) = (x(t), y(t))$.

Remark 3.12. Like the contour integral, this doesn't depend on the choice of parametrization of γ, in the sense of Definition 3.4.

Proposition 3.13 (Interpretation of contour integral as line integrals)**.** *Same setup as in Definition 3.2. Write $f(x,y) = u(x,y) + iv(x,y)$. We have*

$$\int_\gamma f dz = \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy).$$

To remember the above, write $dz = (dx + idy)$ and f in terms of u, v, then expand fdz.

Theorem 3.14 (Green's Theorem)**.** *Let γ be a C^1 curve, oriented counter-clockwise, that is the boundary of an open set $U \subset \mathbb{R}^2$. Let $P, Q : V \to \mathbb{R}$ be C^1 functions, where $V \supset (U \cup \gamma)$ is an open set. Then*

$$\int_\gamma Pdx + Qdy = \iint_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Remark 3.15. This is a special case of the very general *Stokes' Theorem* for a differential form ω: $\int_{\partial U} \omega = \int_U d\omega$. Here ∂ denotes boundary, and d the exterior derivative. One can also think of the above as the two-dimensional version of the *Divergence Theorem*.

3.3 Cauchy's theorem and consequences

Theorem 3.16 (Cauchy's theorem, first version)**.** *Let γ be a simple closed C^1 curve that is the boundary of an open region $U \subset \mathbb{C}$. Let $f : V \to \mathbb{C}$ be a holomorphic function, where $V \supset (U \cup \gamma)$ is an open set. Furthermore, assume that f' is continuous. Then*

$$\int_\gamma f(z)dz = 0.$$

Proof. Using Proposition 3.13, then Green's Theorem, and then the Cauchy-

Riemann equations, we have

$$\begin{aligned}\int_\gamma f dz &= \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy)\\ &= \iint_U \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy + \iint_U \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy\\ &= \iint_U 0dxdy + \iint_U 0dxdy = 0.\end{aligned}$$

■

Example 3.17. Taking $f(z) = \exp(\exp z)$, U to be the disc $D(0,1)$ (and V to be, for instance, $D(0,1.1)$), we get

$$\int_{S^1} \exp(\exp z)dz = 0.$$

A non-example is given by $\int_{s^1} \frac{1}{z}dz = 2\pi i$; the reason that Cauchy's theorem does not apply is that $1/z$ is not holomorphic at 0.

Theorem 3.18 (Deformation)**.** *Let $f : U \to \mathbb{C}$ be holomorphic, and let γ be a simple closed C^1 curve contained in U. Suppose that γ can be continuously deformed within U to γ', also a simple closed C^1 curve in U. Then*

$$\int_\gamma f = \int_{\gamma'} f$$

Proof idea. Assume for simplicity that γ and γ' together form the boundary of a region W (the region that the deformation "traces out"). Connect γ to γ' with a short bridge segment α. We then want to apply Cauchy's theorem to the concatenation $\gamma \cup \alpha \cup \bar{\gamma}' \cup \bar{\alpha}$ (here $\bar{\gamma}$ reverses the orientation of the curve), since this is the boundary of W. (Our version of Cauchy's theorem assumed the boundary was simple, which it is not here; however, by continuity, one can modify it slightly to be simple while only changing the integral by a small amount). The contributions of the integrals over $\alpha, \bar{\alpha}$ cancel out, and we get the desired result. ■

Theorem 3.19 (Cauchy integral formula, simplest version)**.** *Let f be a holomorphic function on an open set containing the closed unit disc centered at 0. Assume that f' is continuous. Then*

$$f(0) = \frac{1}{2\pi i}\int_{S^1} \frac{f(z)}{z}dz,$$

(where S^1 is oriented counter-clockwise).

Proof. Fix $\epsilon > 0$. By continuity of f, we can choose δ such that $|f(0)-f(z)| < \epsilon$ when $|z| < \delta$. Let $S^1(\delta)$ be the circle of radius δ centered at 0. Applying our calculation of $\int_{S^1} dz/z$, the Deformation Theorem, and Lemma 3.11, we get

$$\begin{aligned}
\left| f(0) - \frac{1}{2\pi i}\int_{S^1} \frac{f(z)}{z} dz \right| &= \left| \frac{1}{2\pi i}\int_{S^1} \frac{f(0)-f(z)}{z} dz \right| \\
&= \left| \frac{1}{2\pi i}\int_{S^1(\delta)} \frac{f(0)-f(z)}{z} dz \right| \\
&\leq \frac{1}{2\pi} \operatorname{length} S^1(\delta) \cdot \max_{z \in S^1(\delta)} \left| \frac{f(0)-f(z)}{z} \right| \\
&= \frac{1}{2\pi}(2\pi\delta)(\epsilon/\delta) \\
&= \epsilon.
\end{aligned}$$

Since this holds for any ϵ, we get the desired result. ∎

Simple modifications of the above proof yield:

Theorem 3.20 (Cauchy's integral formula, general form)**.** *Let γ be a simple closed C^1 curve, oriented counter-clockwise, that is the boundary of an open region $U \subset \mathbb{C}$. Let $f : V \to \mathbb{C}$ be a holomorphic function, where $V \supset (U \cup \gamma)$ is an open set. Furthermore, assume that f' is continuous. Then for any $z_0 \in U$,*

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0} dz.$$

The existence of such a formula implies that the values of f inside U are determined completely by the values on the boundary γ.

Theorem 3.21 (Cauchy's formula for derivatives)**.** *Let γ be a simple closed C^1 curve, oriented counter-clockwise, that is the boundary of an open region $U \subset \mathbb{C}$. Let $f : V \to \mathbb{C}$ be a holomorphic function, where $V \supset (U \cup \gamma)$ is an open set. Furthermore, assume that f' is continuous. Then for any $z_0 \in U$,*

$$f^{(k)}(z_0) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)^{k+1}} dz.$$

In particular, all the higher derivatives exist.

Proof. Start from Cauchy integral formula. Then apply Differentiation under the integral sign (see Proposition 3.23 below), with

$$g(z, w) = \frac{f(z)}{z - w},$$

V a small neighborhood of z_0, and U a slight thickening of γ (in particular, U and V are disjoint). ∎

Corollary 3.22 (Infinitely differentiable)**.** *Let $U \subset \mathbb{C}$ open, and $f : U \to \mathbb{C}$ holomorphic such that f' is continuous. Then f is infinitely complex differentiable.*

Below we will consider functions of two complex variables. We will generally think of this as a family of functions of one complex variable, parametrized by a complex number. In below, we will consider continuity of functions from $\mathbb{C}^2 \to \mathbb{C}$. We will also work with *complex partial derivatives*; here one considers fixing one variable.

Proposition 3.23 (Differentiation under integral sign)**.** *Let γ be a C^1 oriented closed curve in $\mathbb{C}$. Let $g : V \times U \to \mathbb{C}$ be a function, with U, V both bounded, open subsets of $\mathbb{C}$, and $U \supset \gamma$. Assume that on the whole domain $g(w, z)$ is continuous and $\frac{\partial}{\partial w} g(w, z)$ exists and is continuous.*

Then $\int_\gamma g(w, z)dz$ is a complex differentiable function on V and

$$\frac{d}{dw} \int_\gamma g(w, z)dz = \int_\gamma \frac{\partial g(w, z)}{\partial w} dz.$$

Proof idea. Show that the difference quotients converge uniformly (using continuity of the partials, and compactness of γ), which allows one to interchange limit and integral. ■

Removing assumption that f' is continuous.

Theorem 3.24 (Goursat)**.** *Let R be a closed solid rectangle, and let f be a holomorphic function on an open set containing R. Then*

$$\int_{\partial R} f(z) = 0.$$

Remark 3.25. This is less general than Cauchy's theorem stated previously, in the sense that the curve has to be very special, the boundary of a rectangle. Note, however, that unlike in that statement, we are *not* assuming that f' is continuous.

Proof. Suppose not. By scaling, we can assume that the longest side of R has length 1, and that $\int_{\partial R} f(z) = 1$. Now divide R into four congruent rectangles, each similar to the original one. The sum of the integrals over these four rectangles equals $\int_{\partial R} f(z)dz$, since the parts over interior segments cancel out. Thus we can choose one of these rectangles, call it R_1 such that $| \int_{\partial R_1} f dz| \geq 1/4$. Subdividing further, for any positive integer n we can find a rectangle R_n, the longer side of which has length 2^{-n}, such that $| \int_{\partial R_n} f dz| \geq 4^{-n}$.

Now the intersection $\cap_n R_n$ is a single point z_0. Pick any $\epsilon > 0$. By the complex differentiability of f at z_0, we have that

$$f(z) = f(z_0) + f'(z_0) \cdot (z - z_0) + c(z) \cdot (z - z_0),$$

where $|c(z)| \le \epsilon$, provided that $z \in R_n$ and n is sufficiently large (depending on ϵ). Then, using that we already know the result for constants and linear functions (by Corollary 3.8, since they have antiderivatives), and the Triangle inequality for integrals (Lemma 3.11), we get

$$\begin{aligned} 4^{-n} &\le \left| \int_{\partial R_n} (f(z_0) + f'(z_0) \cdot (z - z_0) + c_n(z) \cdot (z - z_0))\, dz \right| \\ &= \left| \int_{\partial R_n} c_n(z) \cdot (z - z_0) dz \right| \\ &\le \text{length}(\partial R_n) \cdot \max_{z \in \partial R_n} |c_n(z) \cdot (z - z_0)| \\ &\le \text{length}(\partial R_n) \cdot \max_{z \in \partial R_n} |c_n(z)| \cdot \max_{z \in \partial R_n} |z - z_0| \\ &\le (4 \cdot 2^{-n}) \cdot \epsilon \cdot (2 \cdot 2^{-n}). \end{aligned}$$

Rearranging gives $\epsilon \ge 1/8$, contradicting that we can choose ϵ as small as we wish. ■

Theorem 3.26. *If $U \subset \mathbb{C}$ open, and $f : U \to \mathbb{C}$ is holomorphic, then f' is continuous on U.*

Proof. Since continuity is a local property, we can assume that U is a disc. We first proceed along the lines of proof of Theorem 3.10, defining a function that we will show is the antiderivative of f. Pick $z_0 \in U$. Define

$$F(z) := \int_\gamma f(\zeta) d\zeta,$$

where γ is any path connecting z_0 to z that consists of finitely many horizontal and vertical segments. By repeated application of Goursat's theorem, this is well-defined, i.e. doesn't depend on choice of the horizontal/vertical path. (Here we are using that U is a disc, hence if it contains the boundary of a rectangle, it contains the solid rectangle.) We can then proceed as in proof of Theorem 3.10 to show that $F' = f$.

Now we apply Cauchy's formula for derivatives to F; the hypothesis is satisfied since F' is f, which we know is continuous (since it's differentiable). So we get that F' is differentiable, and in particular continuous. ■

We can now get a version of Cauchy's theorem without knowing a priori that f' is continuous.

Theorem 3.27 (Cauchy)**.** *Let γ be a simple closed C^1 curve that is the boundary of an open region $U \subset \mathbb{C}$. Let $f : V \to \mathbb{C}$ be a holomorphic function, where $V \supset (U \cup \gamma)$ is an open set. Then*

$$\int_\gamma f(z) dz = 0.$$

Proof. Apply the previous theorem to get that f' is continuous. Then we can apply Cauchy's theorem, first version (Theorem 3.16). ■

Remark 3.28. We can similarly drop the assumption of continuity of f' in the Cauchy integral formula, and the Cauchy integral formula for derivatives.

3.4 Applications of complex integration theory

Theorem 3.29 (Cauchy bound)**.** *If f is holomorphic on an open set that contains $D(z_0, R)$ and its boundary, then*

$$|f^{(k)}(z_0)| \leq k! \cdot \frac{\max_{z \in \partial D} |f(z)|}{R^k}.$$

Proof. Apply Triangle inequality for integrals (Lemma 3.11) to Cauchy's formula for derivatives (Theorem 3.21). ■

Definition 3.30. An *entire function* is a function $f : \mathbb{C} \to \mathbb{C}$ that is holomorphic.

(The domain is the *entire* plane).

Theorem 3.31 (Liouville)**.** *Any bounded entire function is constant.*

Proof. Suppose $|f(z_0)| \leq M$ for all $z_0 \in \mathbb{C}$. We apply the Cauchy bound with $k = 1$. For any $z_0 \in \mathbb{C}$, we get that

$$|f'(z_0)| \leq \frac{\max_{z \in \partial D} |f(z)|}{R} \leq \frac{M}{R}.$$

Taking $R \to \infty$, we get that $f'(z_0) = 0$. Since this holds for all z_0, f must be constant (by e.g. the Fundamental theorem of calculus for contour integrals). ■

Theorem 3.32 (Fundamental theorem of algebra)**.** *Consider a complex polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, with $a_n \neq 0$ and $n \geq 1$. Then p has a root in $\mathbb{C}$.*

Proof. Assume the contrary. Then $f(z) := 1/p(z)$ is an entire function. We claim that f is bounded. First note

$$\lim_{z \to \infty} \frac{p(z)}{a_n z^n} = 1,$$

and this implies that $\lim_{z \to \infty} f(z) = 0$. Thus we can choose some large closed ball D such that for $z \notin D$ we have $|f(z)| \leq 1$. Now f is a continuous function, hence on any compact set, in particular on D, it is bounded in magnitude, say by M. But then $|f(z)| \leq \max\{1, M\}$ for all $z \in \mathbb{C}$, i.e. f is bounded. Then Liouville's theorem implies that f is contant, contradiction. Hence p must have a root. ■

The following is a converse to Cauchy's theorem.

Theorem 3.33 (Morera)**.** *Let $U \subset \mathbb{C}$ open and $f : U \to \mathbb{C}$ continuous. Suppose for all C^1 closed curves γ, we have*

$$\int_\gamma f(z)dz = 0.$$

Then f is holomorphic on U.

Proof. The condition implies path-independence of contour integrals of f. So Theorem 3.10 gives that f has an antiderivative F. Then by Corollary 3.22 applied to F, we get that $F' = f$ is itself holomorphic. ■

Remark 3.34. Since being holomorphic is a local property, in the above it suffices to pick for each point $z \in U$ a small disc $D(z, r)$ and prove that the integral over any curve contained in $D(z, r)$ is zero. And in fact, a similar proof yields that it is enough to prove that the integral over any *rectangle* contained in $D(z, r)$ is zero.

Corollary 3.35 (Riemann's removable singularity theorem)**.** *Let $U \subset \mathbb{C}$ be open and $p \in U$. Suppose that $f : U - \{p\} \to \mathbb{C}$ is holomorphic, and that f extends to a continuous function $\tilde{f} : U \to \mathbb{C}$. Then $\tilde{f}$ is holomorphic.*

Proof. Use the local rectangle version of Morera's theorem, discussed in Remark above. If the solid rectangle R does not contain p, then $\int_{\partial R} \tilde{f} = 0$, by Cauchy's theorem. If R contains p, by the Deformation theorem, $\int_{\partial R} \tilde{f} = \int_{\partial R_\epsilon} \tilde{f}$, where R_ϵ is a solid rectangle containing p with longest side ϵ. By continuity of $\tilde{f}$ and the Triangle inequality for integrals, this latter integral tends to zero as $\epsilon \to 0$. ■

Theorem 3.36 (Mean Value Property)**.** *Let $U \subset \mathbb{C}$ open, and $f : U \to \mathbb{C}$ holomorphic. Suppose U contains the closure of some disc $D(z_0, r)$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = f(z_0).$$

Proof. We apply Cauchy's integral formula, using the definition of path integral with the standard parametrization of the circle:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} (rie^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta. \end{aligned}$$

■

Theorem 3.37 (Maximum Modulus Principle)**.** *Let $U \subset \mathbb{C}$ open and connected, and $f : U \to \mathbb{C}$ holomorphic. If for some $p \in U$, $|f(p)| = \sup_{z \in U} |f(z)|$, then f is constant.*

Proof. Since U is open, it contains some ball $D(p, \delta)$. We first show that f is constant on $D(p, r)$ for any $r < \delta$. By the Mean Value Property

$$|f(p)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta}) d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(p + re^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f(p)| d\theta = |f(p)|.$$

Thus equality must hold in each of the two inequalities. From the second inequality being an equality, we see that $|f(p + re^{i\theta})| = |f(p)|$ for every θ (otherwise, by continuity of f, the left would be less than the right by a definite amount on a whole arc, but then the averages could not be equal). And in fact, if the angle of $f(p + re^{i\theta})$ varied, then there would be cancellation in the average, and the first equality above would fail. So we get that f is constant on $\partial D(p, r)$, and since this holds for all $r < \delta$, we get that $|f|$ is constant on any such $D(p, r)$.

Now let $V = \{z \in U : f(z) = f(p)\}$. By the above, we know that V is open. On the other hand, by continuity of f, we also know V is closed. Since U is connected, we must have that $V = U$. ■

All material up to here is fair-game for the prelim.

Question: When does a given holomorphic function admit an antiderivative?

Definition 3.38. A subset $E \subset \mathbb{R}^n$ is said to be *simply connected* if every loop γ in E can be continuously contracted to a point (staying within E the whole time).

Examples: $\mathbb{C}$, $D(z_0, r)$, half-planes
Non-examples: $D(z_0, r)^*$, S^1.

Theorem 3.39. *If $U \subset \mathbb{C}$ is open and simply connected, and $f : U \to \mathbb{C}$ holomorphic, then for any C^1 closed curve γ in C, we have*

$$\int_\gamma f(z) dz = 0.$$

Proof. Since the U is simply connected, γ can be continuously contracted to a point within U. By the Deformation theorem, the contour integral does not change by this deformation. Since the integral over a point is zero, we are done. ■

Theorem 3.40 (Existence of antiderivative)**.** *Let $U \subset \mathbb{C}$ be* simply connected *and open, and $f : U \to \mathbb{C}$ holomorphic. Then f admits an antiderivative: there is a holomorphic function $F : U \to \mathbb{C}$ with $F' = f$.*

Proof. By the previous theorem, contour integrals in U are path independent (to compare integrals along γ and γ', each from z to w, consider $\gamma \cup \bar{\gamma}'$ which is a closed curve, hence the integral over it is zero). Hence by Theorem 3.10, an antiderivative exists. ■

3.5 Harmonic functions

Question: Does the real (or imaginary) part of a holomorphic function $f = u+iv$ have any special properties?

Recall that we have the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Taking the partial of the first equation wrt x, and of the second wrt y gives

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \text{and} \quad \frac{\partial^2 u}{\partial y} = -\frac{\partial^2 v}{\partial y \partial x}.$$

(The existence of these partials is guaranteed by Corollary 3.22). Summing the two equations and using symmetry of mixed partials (Clairaut's theorem) gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0.$$

Definition 3.41. A C^2 (twice differentiable, with continuous second derivative) function $u : U \to \mathbb{R}$, where $U \subset \mathbb{R}^2$ is open, is *harmonic* if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The computation above (and a similar one for imaginary part) yield:

Theorem 3.42. *Let $U \subset \mathbb{C}$ open, and $f : U \to \mathbb{C}$ holomorphic. Then* $\mathrm{Re}(f)$ *and* $\mathrm{Im}(f)$ *are holomorphic functions.*

A converse holds, with some additional condition on U.

Theorem 3.43. *Let $U \subset \mathbb{C}$ be a simply connected open set, and $u : U \to \mathbb{R}$ harmonic. Then there exists a holomorphic function $F : U \to \mathbb{C}$ with* $\mathrm{Re}(F) = u$.

Proof. From our result on Cauchy-Riemann equations, Theorem 2.16, we see that if such an F were to exist, its derivative would equal $f := (\partial u/\partial x) - i(\partial u/\partial y)$. So our guess for F should be the antiderivative of f.

By Fact 2.17, f is real-differentiable (this is why we take u to be C^2 in the definition of harmonic, not just twice differentiable). Then, by a simple computation using symmetry of mixed partials, and the harmonic condition, we see that f satisfies the Cauchy-Riemann equations. Hence by Theorem 2.16, f is holomorphic.

Now Theorem 3.40 on existence of antiderivatives gives us a holomorphic function F with $F' = f$. Now $\mathrm{Re}(F') = \partial \mathrm{Re}(F)/\partial x$, while we also have $\mathrm{Re}(F') = \mathrm{Re}(f) = \partial u/\partial x$. Hence $\partial \mathrm{Re}(F)/\partial x = \partial u/\partial x$. A similar computation yields $\partial \mathrm{Re}(F)/\partial y = \partial u/\partial y$. Together, these imply that $\mathrm{Re}\, F$ and u differ by constant. Translating $\mathrm{Re}(F)$ by this constant gives the desired holomorphic function. ∎

If $u, v : U \to \mathbb{R}$ are functions such that there is a holomorphic function $F : U \to \mathbb{C}$ with $\mathrm{Re}(F) = u$ and $\mathrm{Im}(F) = v$, we say that u, v are *harmonic conjugates.*

Corollary 3.44 (Mean Value Property for harmonic functions)**.** *Let $U \subset \mathbb{R}^2$ be open, and let $u : U \to \mathbb{R}$ be a harmonic function. Suppose that U contains the closure of a disc $D(z_0, r)$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta = u(z_0).$$

Proof. By the previous result, Theorem 3.43, there is a holomorphic function F with $\mathrm{Re}(F) = u$ (we apply that result on a slightly larger disc $D(z_0, r + \delta)$, which becomes the domain of F; the existence of such a disc contained in U follows from compactness of the closure of $D(z_0, r)$, and the assumption that U is open).

We then apply the Mean Value Property (Theorem 3.36) for the holomorphic function F for the disc $D(z_0, r)$, and take real parts to give the desired result. ■

Proposition 3.45 (Maximum principle)**.** *Let $U \subset \mathbb{C}$ open and connected, and $u : U \to \mathbb{C}$ harmonic. If for some $p \in U$, $u(p) = \sup_{z\in U} u(z)$, then u is constant.*

Proof. Argue as in proof of the Maximum Modulus Principle, Theorem 3.37, using the Mean Value Property for harmonic functions. The harmonic case is a little easier, since once one doesn't have to deal with absolute values. ■

Dirichlet problem. Suppose that $U \subset \mathbb{C}$ is a bounded, connected, and open. Let $f : \partial U \to \mathbb{R}$ be a continuous function. The *Dirichlet problem* asks to find a continuous $u : U \cup \partial U \to \mathbb{R}$ such that $u|_U$ is harmonic and $u|_{\partial U} = f$.

One physical situation that the Dirichlet problem models is temperature of a bounded region U at equilibrium. The temperature at the boundary is forced, according to some function f. The equilibrium temperatures will be given by the harmonic solution to the Dirichlet problem on U with boundary data f.

Theorem 3.46. *If a solution exists to the Dirichlet problem for U, f, then it is unique.*

Proof. Suppose that u, v are both solutions. Consider $u - v$. Note that $u - v|_{\partial U}$ is identically 0. Apply the Maximum principle to the harmonic function $u - v|_U$. If $u - v|_U$ is constant, then since it's continuous and its value on ∂U is zero, we must have that $u - v$ is identically zero, i.e. $u = v$, and we're done. Otherwise, we get that $u - v|_U$ does *not* attain a maximum. On the other hand, $\bar{U}$ is compact (since U is bounded), hence $u - v$ achieves a maximum on $\bar{U} = U \cup \partial U$. Since a maximum of $u - v$ is not achieved on U, it must be achieved on ∂U, where its value is 0. This means that $u - v \leq 0$ on all of $\bar{U}$.

On the other hand, we can argue similarly for $v - u$, getting either that this is constant and thus identically 0 (so we're done), or that $v - u \leq 0$ on all of $\bar{U}$.

Together the two inequalities imply $u = v$.

■

4 Infinite Series

4.1 Definitions and basic properties

Does a convergent power series give rise to a holomorphic function? Can any holomorphic function be represented as a power series? The answer to both of these questions is yes! We will see why, and then study some applications.

Definition 4.1 (Convergence of sequences/series)**.** A sequence $z_1, z_2, \ldots \in \mathbb{C}$ is said to *converge* to $z \in \mathbb{C}$ if for every $\epsilon > 0$, there exists an integer N such that if $n \geq N$ then

$$|z_n - z| < \epsilon.$$

A series $\sum_{i=1}^{\infty} z_i$ is said to converge if the partial sums $\sum_{i=1}^{n} z_i$ form a convergent sequence.

Proposition 4.2 (Cauchy criterion)**.** *A sequence $z_1, z_2, \ldots \in \mathbb{C}$ converges to some limit in $\mathbb{C}$ if and only if for all $\epsilon > 0$, there exists N such that if $n, m \geq N$ then*

$$|z_n - z_m| < \epsilon.$$

(Sequences with this property are called Cauchy sequences*.)*

The proof of this, which won't be given here, relies on the *completeness* of the complex numbers (which follows easily from completeness of the real numbers).

Definition 4.3. A series $\sum_{i=1}^{\infty} z_i$ is said to *converge absolutely* if $\sum_{i=1}^{\infty} |z_i|$ is a convergent series.

Using the Cauchy criterion, one sees that absolute convergence implies convergence (the converse is not true, since one can have cancellation).

The most important tool for deciding convergence of series is comparison with a geometric series. There are several manifestations of this, e.g. the root and ratio tests.

Definition 4.4 (Pointwise convergence)**.** Let $E \subset \mathbb{C}$. A sequence of functions $f_n : E \to \mathbb{C}$, $n = 1, 2, \ldots$, is said to *converge pointwise* to $f : E \to \mathbb{C}$ if for all $z \in E$,

$$\lim_{n \to \infty} f_n(z) = f(z).$$

Example: $E = D(0, 1)$, $f_n(z) = z^n$, $f(z) = 0$.

Definition 4.5 (Uniform convergence)**.** Let $E \subset \mathbb{C}$. A sequence of functions $f_n : E \to \mathbb{C}$, $n = 1, 2, \ldots$, is said to *converge uniformly* to $f : E \to \mathbb{C}$ if for any $\epsilon > 0$, there exists an integer N such that if $n \geq N$ then

$$|f(z) - f_n(z)| < \epsilon$$

for all $z \in E$.

The difference between uniform and pointwise convergence is that in pointwise, the rate of convergence is allowed to depend on the point z, while in uniform, the same rate must work for all n. The above example that converged pointwise does not converge uniformly, since as z gets closer to the unit circle, the convergence takes longer and longer.

Proposition 4.6 (Cauchy criterion for uniform convergence)**.** *The sequence f_n converges uniformly iff for all $\epsilon > 0$, there exists an integer N such that if $n, m \geq N$, then*

$$|f_n(z) - f_m(z)| < \epsilon.$$

Proof. Begin by applying Cauchy's criterion for sequences to find a candidate limit function f. Then use the hypothesis to show uniform converge of the f_n to this f. ∎

Theorem 4.7. *The limit of a uniformly convergent sequence of continuous functions is continuous.*

The same proof used in real analysis ("Three ϵ argument") works.

4.2 Sequences of holomorphic functions

Proposition 4.8 (Integration and uniform convergence)**.** *Suppose that γ is a C^1 closed curve. Let $f_n, f : \gamma \to \mathbb{C}$ be functions such that $f_n \to f$ uniformly. Then*

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z)dz.$$

Proof. By uniform convergence, given $\epsilon > 0$, we can choose $N > 0$ such that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N$ and all $z \in \gamma$. Then by the Triangle inequality for integrals,

$$\begin{aligned}\left|\int_\gamma f(z) - \int_\gamma f_n(z)dz\right| &\leq \int_\gamma |f(z) - f_n(z)|dz \\ &\leq \text{length}(\gamma) \cdot \max_{z\in\gamma} |f(z) - f_n(z)| \\ &\leq \text{length}(\gamma) \cdot \epsilon,\end{aligned}$$

which goes to zero as $\epsilon \to 0$. ∎

Theorem 4.9 (Uniform convergence implies holomorphic)**.** *Let $U \subset \mathbb{C}$ and $f_n : U \to \mathbb{C}$ be a sequence of holomorphic functions. If $f_n \to f$ uniformly then f is also holomorphic.*

Remark 4.10. We can replace the condition of uniform convergence with *local* uniform convergence: any z has a neighborhood $B = D(z, \delta)$ for which $f_n|_B$ converge uniformly to $f|_B$.

Proof. By Cauchy's theorem, for each n, $\int_\gamma f_n dz = 0$ for any closed contour γ in B. Uniform convergence implies convergence of contour integrals, hence $\int_\gamma f dz = 0$. Then Morera's theorem gives that f is holomorphic. ∎

Examples showing unif conv of certain geom series: $1 + z + z^2$

Proposition 4.11. *If $f_n \to f$ locally uniformly, then $f_n' \to f'$ locally uniformly.*

Proof. Cauchy integral formula for derivatives. ∎

Proposition 4.12 (Weierstrass M-test)**.** *Let $E \subset \mathbb{C}$ and $f_n : E \to \mathbb{C}$. Suppose there exist $M_n \geq 0$ such that*

- *$|f_n(z)| \leq M_n$ for all $z \in E$, and*
- *$\sum_n M_n$ is convergent.*

Then $\sum_n f_n$ converges absolutely and uniformly to some function $f : E \to \mathbb{C}$.

Proof. Apply Cauchy criterion for uniform convergence, using the fact that the tail $\sum_{n \geq k} M_n$ tends to zero as $k \to \infty$. ∎

Example: $\sum_{n=1}^\infty z^n$ converges absolutely and uniformly on any ball $D(0, \delta)$ with $\delta < 1$; in the above take $M_n = \delta^n$. Similar reasoning applies to show $\sum_{n=1}^\infty n z^n$ converges on any such disc.

Theorem/Definition 4.13. *Given a power series $\sum_{n=1}^\infty a_n z^n$, the* radius of convergence *is defined as the unique number $R \in [0, \infty]$ such that*

1. *If $0 < r < R$ then the series converges absolutely and uniformly on $D(0, r)$, and*
2. *The series diverges at any z with $|z| > R$.*

Proof. Let
$$R := \sup_z \left\{ |z| : \sum_{n=1}^\infty a_n z^n \text{ converges} \right\}$$
To prove 1. we suppose $r < R$. By the definition of r, we can pick w with $|w| > r$ such that the series converges at w, and in particular the terms $a_n w^n$ must be bounded in absolute value by some M. It follows that
$$|a_n r^n| = |a_n| (r/|w|)^n |w|^n \leq M (r/|w|)^n.$$
Applying the Weierstrass M-test with $M_n = M(r/|w|)^n$ then gives that $\sum_n a_n^n$ converges absolutely and uniformly on $D(0, r)$.

Part 2. is immediate from the definition of R. ∎

Arguments similar to those above yield:

Theorem 4.14 (Hadamard)**.** *The radius of convergence of $\sum_n a_n z^n$ is given by*

$$1/\limsup_{n\to\infty} |a_n|^{1/n}.$$

Theorem 4.15. *Let $f(z) = \sum_n a_n z^n$ be a power series with radius of convergence R. Then f defines a holomorphic function on $D(0, R)$. (If $R = \infty$, one should replace $D(0, R)$ by $\mathbb{C}$.)*

Proof. By Theorem/Definition 4.13, the sum converges uniformly on any $D(0, r)$ with $r < R$. Then by Theorem 4.9, we get that f is holomorphic on $D(0, r)$. Since this is true for any $r < R$, we see that f is holomorphic on $D(0, R)$. ■

The above greatly enriches our supply of holomorphic functions: as long as a_n decays fast enough with n, the power series $\sum_n a_n z^n$ will be holomorphic on some disc. Most such functions are not of the type we have been studying up to now (namely polynomials, rational functions, roots, or elementary functions like $\exp, \sin, \cos, \log$).

4.3 Taylor series

Theorem 4.16. *If $f : D(0, R) \to \mathbb{C}$ is holomorphic, its Taylor series*

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

has radius of convergence at least R.

Proof. Apply the Cauchy bound (Theorem 3.29) and Hadamard's theorem. ■

Note that the above does *not* say that the Taylor series actually converges to f; we still need to prove this.

Cautionary example: The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is infinitely differentiable at the origin, and all derivatives are zero there, so the Taylor series converges. But it converges to 0, which does not agree with the function near 0.

Proposition 4.17 (Termwise integration)**.** *Suppose that γ is a C^1 closed curve. Let $f_n, f : \gamma \to \mathbb{C}$ be functions such that $\sum_{n=1}^{\infty}$ converges uniformly to f. Then the series can be integrated termwise, i.e.*

$$\int_\gamma f(z)dz = \sum_{n=1}^{\infty} \int_\gamma f_n(z)dz.$$

Proof. Follows from Proposition 4.8, and linearity of the integral for finite sums. ∎

Theorem 4.18 (Taylor series)**.** *If $f : D(0, R) \to \mathbb{C}$ is holomorphic, its Taylor series*

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

converges to f on $D(0, R)$. The convergence is uniform on $D(0, r)$ for any $r < R$.

Example: $f(z) = e^z$ is equal to $\sum_{n=0}^{\infty} z^n/n!$.

Proof. We begin with the Cauchy integral formula for points $z \in D(0, r)$, with the contour deformed such that its center is at 0:

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(0,r)} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We then observe that the denominator can be manipulated into a form that looks like the formula for a geometric series:

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta(1 - z/\zeta)} = \frac{f(\zeta)}{\zeta} \cdot 1 + \frac{f(\zeta)}{\zeta} \cdot (z/\zeta) + \frac{f(\zeta)}{\zeta} \cdot (z/\zeta)^2 + \cdots .$$

If we take z to be some fixed value in $D(0, r)$, then the above is series of functions of ζ that converges uniformly on $\partial D(0, r)$, by the Weierstrass M-test.

Hence from the expression for f above, and Proposition 4.17 (Termwise integration), we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial D(0,r)} \left(\frac{f(\zeta)}{\zeta} \cdot 1 + \frac{f(\zeta)}{\zeta} \cdot (z/\zeta) + \frac{f(\zeta)}{\zeta} \cdot (z/\zeta)^2 + \cdots \right) d\zeta \\ &= \frac{1}{2\pi i} \left(\int_{\partial D(0,r)} \frac{f(\zeta)}{\zeta} d\zeta + z \int_{\partial D(0,r)} \frac{f(\zeta)}{\zeta^2} d\zeta + z^2 \int_{\partial D(0,r)} \frac{f(\zeta)}{\zeta^3} d\zeta + \cdots \right) \\ &= f(0) + f'(0) z + \frac{f''(0)}{2} z^2 + \cdots , \end{aligned}$$

where we have used Cauchy's formula for derivatives for the last step. ∎

Corollary 4.19. *Any holomorphic function is analytic on its domain.*

Definition 4.20. Given a holomorphic function $f : D(p, R) \to \mathbb{C}$, not identically zero, the *order of vanishing* of f at p is the smallest k such that $f^{(k)}(p) \neq 0$.

The k in the above is always finite, since by Theorem 4.18 (Taylor series), if all the derivatives are zero, then f is identically zero.

Example: the order of vanishing of $z^2 + z^3 + z^4 + \cdots$ at 0 is equal to 2.

Theorem 4.21 (Isolation of zeros)**.** *Let $f : D(p, R) \to \mathbb{C}$ holomorphic, and suppose f is not identically zero. If $f(p) = 0$, then there exists some ball $B \subset U$, $B \ni p$ such that $f(z) \neq 0$ for all $z \in B - p$.*

Proof. By Theorem 4.18 (Taylor series), for $z \in D(p, R)$, we have

$$f(p) = \sum_{n=0}^{\infty} a_n (z-p)^n = (z-p)^k \left(a_k + a_{k+1}(z-p) + a_{k+2}(z-p)^2 + \cdots \right),$$

where $a_k \neq 0$ (so k is the order of vanishing). The series $a_k + a_{k+1}(z-p) + a_{k+2}(z-p)^2 + \cdots$ has the same radius of convergence as $\sum_{n=0}^{\infty} a_n (z-p)^n$, since the ratios of successive terms are the same. Hence this series defines a holomorphic function $g(z)$ on $D(p, r)$, and we have $f(z) = (z-p)^k g(z)$. Note that $g(p) = a_k \neq 0$, and since g is holomorphic, hence continuous, we have $g(z) \neq 0$ on some ball $B = D(p, \delta)$. Then $f(z) \neq 0$ for $z \in B - p$. ∎

Analytic continuation. Consider the holomorphic function $f : D(0, 1) \to \mathbb{C}$ given by the convergent power series $f(z) = 1 + z + z^2 + \cdots$. The radius of convergence is 1, so the series does not converge on any circle centered at the origin of radius bigger than 1. And there is no holomorphic function g defined on a bigger ball $D(0, R)$ that coincides with f on $D(0, 1)$, since if so we could apply the theorem on Taylor series to this g to get a convergent series on $D(0, R)$ that would coincide with the original series. Of course, we know that $f(z) = \frac{1}{1-z}$, and this function is undefined at 1. However this formula for f gives a well defined holomorphic function on $\mathbb{C} - \{1\}$, i.e. the original function has an *analytic continuation.*

A similar story holds for $f(z) = 1 - z^2 + z^4 - z^6 + \cdots$. The Taylor series for f centered at 0 will have radius of convergence 1. Note that $f(z) = 1/(1 + z^2)$. The obstruction to analytically continuing f to a function to $D(0, R)$ for some $R > 1$ is the singularities at $i, -i$. These do not lie on the real line, so the complex plane provides a satisfactory answer for why the Taylor series does not converge past the unit circle, not visible from the real perspective.

Theorem 4.22 (Uniqueness of analytic continuation)**.** *Suppose $U \subset \mathbb{C}$ connected, and $f, g : U \to \mathbb{C}$ holomorphic functions. If f and g are equal on some open ball $B \subset U$, then f and g are equal on all of U.*

Proof. We will show $h := f - g$ is identically zero. Let $U' = \{z : h \text{ is zero near } z\}$, which is open. If w is in the closure of U', pick some ball B around w contained in U. Note that any ball centered at w contains infinitely many zeros of h. So by Isolation of zeros, h is identically zero on some ball centered at w, hence $w \in U'$. So U' is open and closed, hence $U' = U$, since U is connected. ∎

4.4 Laurent series

Given a function that has a singularity, can we still expand it in some sort of series centered at the singularity? For instance, how could we represent

$f(z) = 1/z$ in terms of a series centered at 0? One answer is to allow negative powers of z, so the series for $f(z)$ would just be $1/z$.

In general, we will consider functions defined on *annuli*. For concreteness, we center at 0. For $0 \le r < R \le \infty$, define

$$A_{r,R} := \{z \in \mathbb{C} : r < |z| < R\}.$$

The boundaries (when r, R are non-zero and finite) are the two circles $S^1(r)$ and $S^1(R)$. As a convention we orient these counter-clockwise.

Theorem 4.23 (Laurent series)**.** *Let $f : A_{r,R} \to \mathbb{C}$ be a holomorphic function. Then we can write*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where the series converges absolutely and uniformly on $A_{r',R'}$, for any r', R' with $r < r' < R' < R$. Furthermore, the series with these convergence properties is unique.

Proof. We can express $f(z)$ in terms of the values of f on a small circle centered at z using the Cauchy integral formula. Deforming the contour as much as possible, we get, for all $z \in A_{r',R'}$

$$f(z) = \frac{1}{2\pi i}\int_{S^1(R')} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i}\int_{S^1(r')} \frac{f(\zeta)}{z - \zeta} d\zeta.$$

Now we proceed as in the proof of Taylor's series theorem using the geometric series representations

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\left(1 + (z/\zeta) + (z/\zeta)^2 + \cdots\right),$$
$$\frac{1}{z - \zeta} = \frac{1}{z}\left(1 + (\zeta/z) + (\zeta/z)^2 + \cdots\right).$$

For fixed z, the first series converges uniformly for ζ ranging over $S^1(R')$, while the second converges uniformly for ζ ranging over $S^1(r')$. Thus we can substitute these formulas in the integral expression for f, and integrate termwise to get

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\int_{S^1(R')} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta\right) z^n + \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\int_{S^1(r')} f(\zeta)\zeta^n d\zeta\right) z^{-n-1}.$$

We can deform one of the contours in the last expression above to the same circle $S(r')$ (for $r < r' < R$) to get a uniform formula that covers Laurent coefficients for both negative and positive powers:

$$a_n = \frac{1}{2\pi i}\int_{S^1(r')} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta.$$

For uniqueness, take any other series representation $\sum_n b_n z^n$, multiply by z^m, and integrate termwise; this picks out the b_{-1-m} term, since $\int_S z^k dz = 0$ for $k \ne -1$ and any circle S centered at 0. ■

Example 4.24. Take $f(z) = \cos z/z$ on the punctured plane $A_{0,\infty} = \mathbb{C}^*$. We write the Taylor series for $\cos z$, and divide:

$$f(z) = (1/z)\left(1 - z^2/2! + z^4/4! - \cdots\right)$$
$$= \frac{1}{z} - z/2! + z^3/3! - \cdots .$$

This converges uniformly on $A_{\epsilon,1}$ for any $\epsilon > 0$, by the Weierstrass M-test. Hence by the uniqueness statement in the above theorem, this must be the Laurent series.

Example 4.25. Take $f(z) = e^{1/z}$ on the punctured disk $A_{0,1} = D(0,1)^*$. We can substitute $1/z^2$ into the Taylor series $\sum_{n=0}^{\infty} z^n/n!$ for e^z to get

$$f(z) = \sum_{n=0}^{\infty} z^{-n}/n!$$

which converges uniformly on $A_{\epsilon,1}$ for any $\epsilon > 0$, by the Weierstrass M-test. Hence by the uniqueness statement in the above theorem, this must be the Laurent series.

4.5 Singularities

Suppose $U \subset \mathbb{C}$ open, $p \in U$, and $f : U - \{p\} \to \mathbb{C}$ holomorphic. We can find a small disk in U centered at p, and apply the Laurent expansion to understand how f is behaving at p. For concreteness, we will work with the disk $D(0,1)$.

Theorem/Definition 4.26 (Classification of isolated singularities)**.** *Let $f : D(0,1)^* \to \mathbb{C}$ holomorphic, and consider its Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Let $N = \inf\{n : a_n \neq 0\}$ (if all a_n are zero, take $N = \infty$; note also that we can have $N = -\infty$). Then

1. *If $N \geq 0$, then the series can be extended over 0 to a holomorphic function; in this case 0 is called a* removable singularity

   (a) *if $N = \infty$, f is identically zero*

   (b) *if $\infty > N > 0$ then f is said to have a* zero of order N at 0

   (c) *if $N = 0$, $f(0) \neq 0$.*

2. *If $N < 0$, then f does not extend to a holomorphic function on $D(0,1)$ (since then it would have a Taylor series); the point 0 is a non-removable singularity.*

   (a) *if $-\infty < N < 0$ then f is said to have a* pole at 0 of order $-N$.

(b) if $N = -\infty$, then f is said to have a essential singularity at 0*.*

Definition 4.27. If $U \subset \mathbb{C}$ open and $f : U \to \mathbb{C} \cup \{\infty\}$ is a function that is holomorphic away from a set of isolated singularities which are all poles, then we say that f is a *meromorphic function* on U.

Theorem 4.28 (Riemann's removable singularity theorem, strengthened)**.** *Let $f : D(0,1)^* \to \mathbb{C}$ holomorphic and* bounded*. Then f has a removable singularity at* 0*.*

Proof. Pick $\delta > 0$ and define $g : D(0,\delta) \to \mathbb{C}$ by

$$g(z) = \int_{\partial D(0,\delta)} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Note that g is holomorphic, by Differentiation under the integral sign. We claim that g agrees with f on $D(0,\delta)^*$. For fixed $z \in D(0,\delta)^*$, and any small $\delta' > 0$ we have by Cauchy's integral formula (and Deformation theorem) that

$$f(z) = \int_{\partial D(0,\delta)} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\partial D(0,\delta')} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Since f is assumed bounded on $D(0,\delta)$, and by the triangle inequality for integrals, the integral on the right goes to 0 as $\delta' \to 0$. Thus g and f agree on $D(0,\delta)^*$, and so g can be used to extend f holomorphically over 0. ∎

Theorem 4.29 (Casorati-Weierstrass)**.** *Let $f : D(0,1)^* \to \mathbb{C}$ be a holomorphic function with an essential singularity at* 0*. Then $f(D(0,1)^*)$ is a dense subset of $\mathbb{C}$.*

Proof. Suppose for the sake of contradiction that there is some $p \in \mathbb{C}$ with $p \notin \overline{f(D(0,1)^*)}$. Then there is some ball $D(p,\delta) \subset \mathbb{C}$ that is disjoint from $f(D(0,1)^*)$. Now consider the holomorphic function $g : B(0,1)^* \to \mathbb{C}$,

$$g(z) := \frac{1}{f(z) - p}.$$

Since $|f(z) - p| \geq \delta$ for all z, we see that g is bounded. By Riemann's removable singularity theorem (Theorem 4.28), we get that g extends to a holomorphic function on $D(0,1)$.

Now using the Taylor series for f centered at 0, we can write $g(z) = z^k h(z)$, where h is holomorphic and $h(0) \neq 0$. Near 0, we then get that

$$f(z) - p = \frac{1}{g(z)} = z^{-k} \frac{1}{h(z)},$$

so at $z = 0$, f has a pole of order k (or is holomorphic). But we assumed it had an essential singularity here, contradiction. ∎

Example: $e^{1/z}$ (think about e^z near infinity)

5 Residue theorem and applications

5.1 Residues and the residue theorem

Definition 5.1 (Residue)**.** Let $U \subset \mathbb{C}$ open, and suppose $f : U \to \mathbb{C}$ has an isolated singularity at $p \in U$. The *residue* of f at p, denoted $\operatorname{Res}(f; p)$, is defined to be the a_{-1} term of the Laurent series for f on small punctured disk centered at p.

From our proof of the Laurent series theorem, we see that

$$\operatorname{Res}(f; p) = \frac{1}{2\pi i} \int_\gamma f(z) dz,$$

where γ is a small loop centered at p that does not enclose any other singularities of f.

Theorem 5.2 (Residue Theorem)**.** *Let $U \subset \mathbb{C}$ open, $P \subset U$ a finite set, and $f : U - P \to \mathbb{C}$ holomorphic. Suppose that $\gamma \subset U - P$ is a C^1 closed curve that bounds a region $V \subset U$. Then*

$$\int_\gamma f(z) dz = 2\pi i \sum_{p \in P \cap V} \operatorname{Res}(f; p).$$

Proof. Let $p_1, \ldots, p_n$ be the points in $P \cap V$. Let $B_i \subset U$ be a small disc centered at p_i that does not contain any of the other p_j. Orient all boundaries counter-clockwise. Cauchy's theorem and the integral expression for residues give

$$\int_\gamma f(z) dz = \sum_i \int_{\partial B_i} f(z) dz = 2\pi i \sum_i \operatorname{Res}(f; p_i).$$

■

Example 5.3. Compute

$$\int_{S^1(2)} \frac{1}{1+z^2} dz.$$

We need to find the residues of $f(z) = 1/(1+z^2)$ at i and $-i$. To compute the residue at i, write

$$\begin{aligned} f(z) &= \frac{1}{(z-i)(z+i)} = \frac{1}{z-i}\left(a_0 + a_1(z-i) + a_2(z-i)^2 + \cdots\right) \\ &= a_0(z-i)^{-1} + a_1 + a_2(z-i) + \cdots . \end{aligned}$$

Here $a_0, a_1, \ldots$ are the Taylor coefficients for $1/(z+i)$ centered at i (note that $1/(z+i)$ is holomorphic near i). In particular, $a_0 = 1/(i+i) = -i/2$. So $\operatorname{Res}(f; i) = -i/2$. A similar computation yields $\operatorname{Res}(f; -i) = i/2$. Then the Residue Theorem gives

$$\int_{S^1(2)} \frac{1}{1+z^2} dz = 2\pi i \left(\operatorname{Res}(f; i) + \operatorname{Res}(f; -i)\right) = 2\pi i(-i/2 + i/2) = 0.$$

Example 5.4. Compute

$$\int_{S^1(0.1)} \frac{1}{\sin z} dz.$$

We first compute the residue at 0 by noting that

$$\frac{1}{\sin z} = \frac{1}{z - z^3/3! + z^5/5! - \cdots} = \left(\frac{1}{z}\right)\left(\frac{1}{1 - z^2/3! + z^4/5! - \cdots}\right).$$

The factor on the right is a holomorphic function whose value at 0 is 1, so $\text{Res}(1/\sin(z); 0) = 1$. Then the Residue Theorem gives

$$\int_{S^1(0.1)} \frac{1}{\sin z} dz = 2\pi i \cdot \text{Res}(1/\sin(z); 0) = 2\pi i.$$

5.2 Definite integrals using residue theory

We can evaluate certain purely real definite integrals using by computing a related contour integral.

Example 5.5. Compute

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$$

We take a contour consisting of the interval $I_R = [-R, -R]$ together with a half-circle γ_R in the upper-half plane connecting $R, -R$. By the Residue Theorem

$$\int_{I_R \cup \gamma_R} \frac{1}{1+z^2} dz = 2\pi i \cdot \text{Res}(1/(1+z^2), i) = 2\pi i(-i/2) = \pi.$$

As $R \to \infty$, the integral over γ_R approaches 0 (using Triangle inequality for integrals). Hence

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \lim_{R \to \infty} \int_{I_R} \frac{1}{1+z^2} dz = \pi.$$

Example 5.6. Compute

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx.$$

We use the same contour as in the previous example; this time it contains two poles, at ζ, ζ^3, where ζ is a primitive 8th root of unity. Take $f(z) = 1/(1+z^4)$ and $g(z) = 1+z^4$. Since g has a simple zero at ζ, $\text{Res}(f;\zeta) = 1/g'(\zeta) = 1/(4\zeta^3)$. Similarly, we get $\text{Res}(f;\zeta^3) = 1/(4\zeta)$. Arguing as above we get that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \lim_{R \to \infty} \int_{I_R} \frac{1}{1+z^4} dz = 2\pi i \left(\text{Res}(f;\zeta) + \text{Res}(f;\zeta^3)\right) = \frac{\pi}{\sqrt{2}}.$$

The technique used above works well for integrals of rational function $\int_{-\infty}^{\infty} (P(x)/Q(x)) dx$, where $\deg Q \geq \deg P + 2$, and Q is a polynomial we can factor (and whose roots don't lie on real axis).

Example 5.7. Compute

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta},$$

where $0 < a < 1$. Make the change of variable $z = e^{i\theta}$, so $dz = ie^{i\theta}d\theta$. Note $\cos\theta = (z + 1/z)/2$. So we get

$$I = \int_\gamma \frac{idz}{a(z-a)(z-1/a)}.$$

This has simple poles at $z = a, 1/z$, with residues $i/(a^2-1)$ and $i/(1-a^2)$; only the a lies in the unit disc. Thus by the Residue Theorem,

$$I = 2\pi i \cdot \mathrm{Res}\left(\frac{i}{a(z-a)(z-1/a)}; a\right) = 2\pi i \cdot i/(a^2-1) = 2\pi/(1-a^2).$$

Example 5.8. Compute

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx.$$

Take a closed "Pacman" contour γ consisting of:

- γ_1 a segment of length R starting near origin at angle ϵ to positive x-axis,
- γ_3 a length R ray ending near the origin at angle $-\epsilon$ to postitive x-axis,
- γ_2 a large circular arc of circle centered at 0 connecting the end of γ_1 to end of start of γ_3,
- γ_4 a small circular arc of radius ϵ centered at origin, connecting the start of γ_1 to end of γ_2.

Take the standard ("positive") branch of $f(z) = z^{1/2}/(1+z^2)$ on the region U bounded by γ. Going around the circle takes us to the other ("negative") branch, so

$$\int_0^R \frac{\sqrt{x}}{1+x^2}dx = \lim_{\epsilon\to 0}\int_{\gamma_1} f(z)dz = \lim_{\epsilon\to 0}\int_{-\gamma_3} -f(z)dz = \lim_{\epsilon\to 0}\int_{\gamma_3} f(z)dz.$$

For large R, f has poles at $\pm i$ in U. If g, h are holomorphic with $g(p) \neq 0$ and h has a simple zero at p, then $\mathrm{Res}(g/h; p) = g(p)/h'(p)$. Hence

$$\mathrm{Res}(f; i) = \frac{i^{1/2}}{2(i)} = \frac{1}{2i}\cdot\frac{1+i}{\sqrt{2}},$$

where we have been careful to use the correct branch to compute $i^{1/2}$. Similarly, $\mathrm{Res}(f; i) = \frac{1}{-2i}\cdot\frac{-1+i}{\sqrt{2}}$.

Now applying the Residue Theorem gives

$$\int_\gamma f(z)dz = 2\pi i\left(\mathrm{Res}(f; i) + \mathrm{Res}(f; -i)\right) = 2\pi/\sqrt{2}.$$

Taking limits and expressing the integral in terms of its four parts gives

$$\begin{aligned}2\pi/\sqrt{2} &= \lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz\right) \\ &= 2\lim_{R\to\infty}\int_0^R \frac{\sqrt{x}}{1+x^2}dx + \lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\int_{\gamma_2} f(z)dz + \int_{\gamma_4} f(z)dz\right)\end{aligned}$$

Estimating the γ_2, γ_4 contributions using triangle inequality for integrals give that those terms go to zero. So we get

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx = \pi/\sqrt{2}.$$

5.3 The argument principle

(See Marsden-Hoffman Section 6.2)

Given a holomorphic function f vanishing at p, we denote by $\mathrm{Mult}(f;p)$ the order of the zero at p, assuming f is not identically zero in any neighborhood of p. If f does not vanish at p, we set $\mathrm{Mult}(f;p) = 0$. (Recall the order was defined in Theorem/Definition 4.26.)

Theorem 5.9. *Let $U \subset \mathbb{C}$ open, and $f : U \to \mathbb{C}$ holomorphic. Suppose that $\gamma \subset U$ is a C^1 closed curve that bounds a region $V \subset U$. Suppose that f is non-zero everywhere on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{p\in V}\mathrm{Mult}(f;p).$$

Proof. First note that $\mathrm{Res}(f'/f;p) = \mathrm{Mult}(f;p)$; this is proved by writing $f(z) = (z-p)^k g(z)$, where g is holomorphic and non-zero near p. Then the result follows directly from the Residue Theorem applied to the function f'/f. ∎

Interpretation: Note that $f'/f = \frac{d}{dz}\log f$. Thus one can think of the integral above as $\int_\gamma d\log f$. On domains encircling 0, log is not a well-defined function; however its derivative is, and one can think of the integral as measuring the difference in values for different branches of log. We know that as we go around a circle centered at 0, log increases by $2\pi i$. Thus the integral measures the number of times $f(\gamma)$ winds around the origin. This is related to the number of zeros that f has in the region bounded by γ.

Definition 5.10 (Open map)**.** Let $U \subset \mathbb{C}$ open. A function $f : U \to \mathbb{C}$ is *open* if for any open set $V \subset U$, $f(V)$ is also open.

In other words, by perturbing the input a small amount, the output can be perturbed by a small amount in any direction.

Example: $f(z) = z^2$

Non-examples: $f(z) = |z|$, $f(z) = 0$.

Theorem 5.11 (Open Mapping)**.** *Let $U \subset \mathbb{C}$ open. If $f : U \to \mathbb{C}$ is holomorphic and not locally constant, then it is open.*

Proof. It suffices to show that for any small disc $D(p,r) \subset U$, the image $f(D)$ is open. The idea is to apply the argument principle to the function $f(z) - w$ for various w near $f(p)$. By isolation of zeros, making r smaller we can assume that the only zero of $f(z) - f(p)$ on $\overline{D(p,r)}$ is at $z = p$. The argument principle then gives

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial D(p,r)} \frac{(f(z)-f(p))'}{f(z)-f(p)}dz &= \sum_{p' \in D(p,r)} \mathrm{Mult}(f(z)-w;p') \\ &= \mathrm{Mult}(f(z)-f(p);p) = k,\end{aligned}$$

for some integer $k > 0$. Now for any w, another application of argument principle gives

$$\frac{1}{2\pi i}\int_{\partial D(p,r)} \frac{(f(z)-w)'}{f(z)-w}dz = \sum_{p' \in D(p,r)} \mathrm{Mult}(f(z)-w;p').$$

Now this expression various continuously in w for w near $f(p)$. On the other hand, the right-hand side is always an integer! This means that the expression is locally constant as a function of w. Since it's value is k at $w = f(p)$, it must also equal k for all w near $f(p)$. That means that one of the terms in the sum of multiplicities is positive, i.e. $f(z) = w$ for some z.

■

The open mapping theorem can be used to give a different simple proof of the maximum modulus principle.

5.4 Local Mapping

Theorem 5.12 (Local mapping)**.** *Let $U \subset \mathbb{C}$ open, $f : U \to \mathbb{C}$ holomorphic, and $p \in U$. There is some disc $D(p,r)$, a non-negative integer k, and an injective holomorphic function $g : D(p,r) \to \mathbb{C}$ such that $f(z) = a + g(z)^k$ for all $z \in D(p,r)$, $g(p) = 0$, and $g'(p) \neq 0$.*

Proof. On any small disc about p we can write a Taylor series for f:

$$\begin{aligned}f(z) &= a_0 + a_1(z-p) + a_2(z-p)^2 + \cdots \\ &= a_0 + (z-p)^k\left(a_k + a_{k-1}(z-p) + \cdots\right),\end{aligned}$$

where $a_k \neq 0$. On a possibly smaller disc, $a_k + a_{k-1}(z-p) + \cdots$ is nowhere zero, so it admits a holomorphic kth root function. So $f(z) = a_0 + ((z-p)h(z))^k$. We take $g(z) = (z-p)h(z)$, and note that $g'(p) = h(p) \neq 0$, so g is locally invertible by the inverse function theorem.

■

Corollary 5.13. *A holomorphic function $f : U \to \mathbb{C}$ is locally injective at $z \in U$ iff $f'(z) \neq 0$.*

Recalling the definition of conformal, the above means that conformal is equivalent to holomorphic and locally injective.

6 Automorphisms and conformal mapping

(See Marsden-Hoffman Ch. 5.2 and Ch. 5.3)

6.1 Automorphisms and Mobius transformations

An *automorphism* of U is a holomorphic map $f : U \to U$ that is a bijection. Note that the inverse of a holomorphic bijection is holomorphic, by Corollary 5.13 and the inverse function theorem. (In particular the inverse is continuous. Compare to the setting of continuous functions – one can have a continuous bijection whose inverse is not continuous, for instance the natural map from $[0, 1)$ to S^1).

Theorem 6.1 (Automorphisms of $\mathbb{C}$)**.** *The automorphisms of $\mathbb{C}$ are exactly the maps of the form $z \mapsto az + b$, for some $a, b \in \mathbb{C}$, $a \neq 0$.*

Proof. Let f be such an automorphism. Consider $g : \mathbb{C}^* \to \mathbb{C}$, $g(z) = 1/f(1/z)$. The preimage $f^{-1}(\overline{B(0, 1)})$ is compact (since f is a homeomorphism it preserves compact sets), hence bounded. It follows that $|g(z)| \leq 1$ for z inside some small ball. Thus by the Riemann removable singularity theorem, g extends to a holomorphic function on $\mathbb{C}$. Thus, locally near 0, we can write $g(z) = z^k h(z)$, where h is holomorphic and $h(z) \neq 0$. It follows that $|h(z)| \geq \epsilon$ near $z = 0$ for some $\epsilon > 0$. Hence $|g(z)| \geq \epsilon |z|^k$. Translating this back into information about f gives

$$|1/f(1/z)| \geq \epsilon |z|^k$$
$$|f(1/z)| \leq (1/\epsilon) \frac{1}{|z|^k}.$$

Hence taking $w = 1/z$ we get that $|f(w)| \leq (1/\epsilon)|w|^k$ for all w outside a large ball. Applying this with the Cauchy bound (in a very similar way to HW 8, Problem 4) gives that f is a polynomial. But polynomials of degree larger than 1 are non-injective, hence f must have degree 1. ■

Can we interpret $f(z) = 1/z$ as the automorphism of something? Note that this $f : \mathbb{C}^* \to \mathbb{C}^*$ extends to a continuous function $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by setting $f(\infty) = 0$ and $f(0) = \infty$. And in fact it is "holomorphic" even at 0 and ∞, in an appropriate sense (to define this, one looks at $f(1/z)$ for z large and $1/f(z)$ for z close to 0).

Definition 6.2. A Mobius transformation (aka fractional linear transformation) is a function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the form

$$f(z) = \frac{az+b}{cz+d}.$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

Theorem 6.3. *The Mobius transformations form a* group *under composition. That is, the composition of two Mobius transformations is a Mobius transformation, and every Mobius transformation has an inverse.*

Proof. We set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and define a map $p : \mathbb{C}^2 \to \widehat{\mathbb{C}}$ given by $(z, w) \mapsto z/w$ and note that

$$f(z) = p\left(A(z \quad 1)^T\right).$$

Note that since we are applying p at the end, we can get the same answer by replacing $(z \quad 1)$ with $(zw \quad w)$ for any $w \in \mathbb{C}^*$. Take another Mobius transformation g, which corresponds to some matrix B. Then

$$\begin{aligned} g \circ f &= p\left(B\left(p\left(A\left(z \quad 1\right)^T\right) \quad 1\right)^T\right) \\ &= p\left(B\left(A(z \quad 1)^T\right)\right). \end{aligned}$$

This is the Mobius transformation corresponding to the product matrix BA.

We can then easily see that the inverse of f is the Mobius transformation corresponding to A^{-1} (we are assuming $ad - bc \neq 0$, so the inverse matrix exists). ■

Corollary 6.4. *A Mobius transformation gives a bijection $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. It is holomorphic in the appropriate sense, hence an automorphism.*

The intuition for the following is that Mobius transformations have three complex degrees of freedom (there are 4 complex parameters a, b, c, d, but rescaling them all by the same complex number leads to the same Mobius transformation).

Theorem 6.5 (Triple transitivity)**.** *Given points $z_1, z_2, z_3, w_1, w_2, w_3 \in \widehat{\mathbb{C}}$ with $z_i \neq z_j$ for $i \neq j$ and $w_i \neq w_j$ for $i \neq j$, there exists a unique Mobius transformation f with $f(z_i) = w_i$ for $i = 1, 2, 3$.*

Proof. First we show that we can find a Mobius transformation g with $g(z_1) = 0$, $g(z_2) = 1$, $g(z_3) = \infty$. One easily checks that

$$g(z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

does the job.

Similarly, we can find h with $h(w_1) = 0$, $h(w_2) = 1$, $h(w_3) = \infty$. Then $f = h^{-1} \circ g$ has the desired property.

■

Theorem 6.6. *Any automorphism f of $\widehat{\mathbb{C}}$ is a Mobius transformation.*

Proof. By Triple transitivity, we can compose f with a Mobius transformation g such that $g \circ f(\infty) = \infty$. So $g \circ f$ is an automorphism of $\mathbb{C}$, hence affine, by Theorem 6.1. In particular $g \circ f$ is Mobius, and composing with g^{-1} gives that f is as well. ■

Theorem 6.7 (Circles/lines)**.** *If $S \subset \widehat{\mathbb{C}}$ is a circle or line, and f is Mobius, then $f(S)$ is also a circle or a line.*

Note that f might take a circle to a line.

Proof. Any Mobius transformation is a composition of a sequence of Mobius transformations that are either affine or $z \mapsto 1/z$. Affine maps clearly preserve circles/lines. One can check explicitly that $z \mapsto 1/z$ also does, using the equation for circles and lines in (x, y) coordinates.

There is a also a more synthetic way to show that $z \mapsto 1/z$ preserves circles/lines. First observe that is suffices to show this property for $I(z) = 1/\bar{z}$. In polar coordinates, this map is particularly simple: it sends $r(e^{i\theta}) \mapsto (1/r)e^{i\theta}$, and is an example of an *inversion*. Now one notices that for any $a, b \in D(0, 1)$, the two triangles $a0b$ and $I(b)0I(a)$ are similar. If follows that I takes any circle passing through 0 and lying in the disc to a line. Judicious use of this similarity principle yields the result for all circles/lines. ■

6.2 Conformal mapping

Definition 6.8. Domains $U, V \subset \mathbb{C}$ (or in $\widehat{\mathbb{C}}$) are said to *conformally isomorphic* (or *biholomorphic*) if there exists a holomorphic bijection $U \to V$.

Theorem 6.9. *The upper-half plane $\mathbb{H} := \{z : \text{Im}(z) > 0\}$ and the disc $D(0, 1)$ are conformally isomorphic.*

Proof. We claim that the Mobius transformation $f(z) = \frac{z-i}{z+i}$ restricts to a holomorphic bijection $\mathbb{H} \to D(0, 1)$. In fact, the map takes $(-1, 0, 1)$ to $(i, -1, -i)$. By the circles/lines property, the real line must get mapped to unit circle. Let g be inverse of f; note that g takes the unit circle to $\mathbb{R}$. We will be done if we can show $g(D(0, 1)) = \mathbb{H}$. By connectedness of $D(0, 1)$, to check that $g(D(0, 1)) \subset \mathbb{H}$, it now suffices to check that some point in $\mathbb{H}$ maps into $D(0, 1)$; for instance note that $g(0) = i$.

All that remains is to show that $g(D(0, 1)) \supset \mathbb{H}$. Since $\mathbb{H}$ is connected, it suffices to show that $g(D(0, 1))$ is both open and closed as subset of $\mathbb{H}$. Openness is immediate, since $g(D(0,1))$ is open in $\widehat{\mathbb{C}}$. Note that $g(\overline{D(0,1)})$ is closed in $\widehat{\mathbb{C}}$, since $\overline{D(0,1)}$ is. Using this and the fact that $g(S^1) = \mathbb{R}$, we get that

$$g(D(0,1)) = g(\overline{D(0,1)}) \cap \mathbb{H},$$

is closed in $\mathbb{H}$, and we're done.

■

One can also check that the restriction of the map f in the proof above to the open first quadrant has image the lower half semi-disk.

Other confromal isomorphisms.

Theorem 6.10. *The open first quadrant $Q := \{z : \mathrm{Re}(z), \mathrm{Im}(z) > 0\}$ and the upper-half plane $\mathbb{H}$ are conformally isomorphic via the map $z \mapsto z^2$.*

Proof. The inverse is given by (the standard branch of) $\sqrt{z}$; note that this is well-defined on $\mathbb{H}$. ■

All material up to here is fair-game for the final (any further changes to the document will only be minor corrections/clarifications).

Theorem 6.11. *The automorphisms of $\mathbb{H}$ are exactly the Mobius transformations of the form*

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{R}$, and $ad - bc = 1$.

Examples: $z \mapsto 2z$, $z \mapsto z + 1$.
These are of central importance in (plane) hyperbolic geometry.

Theorem 6.12. *The disc $D(0,1)$ and the complex plane $\mathbb{C}$ are not conformally isomorphic.*

Proof. Suppose $f : \mathbb{C} \to D(0,1)$ were a conformal bijection. By Liouville's theorem, f is constant. But then it cannot be a bijection, since $D(0,1)$ has more than point, contradiction. ■

Theorem 6.13 (Riemann mapping theorem)**.** *Let $U \subset \mathbb{C}$ be open, connected, and simply connected, with $U \neq \mathbb{C}$. Then U is conformally isomorphic to $D(0,1)$.*

There are examples where the boundary is very bad. In the case of polygons, there is a somewhat explicit formula for the map, called the *Schwarz-Christoffel formula.*

6.3 Applications (not on exams)

Dirichlet problem. The Dirichlet problem for the disc can be solved using several different methods. Recall that we specify a continuous function $f : S^1 \to \mathbb{R}$ and want to extend it to a function u on the closed disc that is harmonic on the open disc.

- One approach starts with the observation that the functions z^n and $\bar{z}^m$ on S^1, for n, m non-negative integers, extend naturally to functions on the disc, whose real and imaginary parts are each harmonic. Any continuous function $f : S^1 \to \mathbb{C}$ can be written as an (infinite) linear combination of these functions (this is not obvious: the key idea is that one can approximate a "δ function" with a sharp peak at 0 by trigonometric polynomials, and then integrate this against f). Then since finite linear combinations of $z^n, \bar{z}^m$ can be extended, a limiting argument gives that any f can also be extended harmonically.

  One can adapt the existence argument above to get an actual formula for the extension u. In fact, the coefficients in the expression for f in terms of $z^n, \bar{z}^m$ are closely related to Laurent coefficients (since $\bar{z} = 1/z$ on the unit circle). Using the formula for Laurent coefficients, one gets a formula for u:
  $$u(z) = \frac{1}{2\pi} \int_0^{2\pi} K(z, e^{i\theta}) f(e^{i\theta}) d\theta,$$
  where $K(z, e^{i\theta}) := \frac{1-|z|^2}{|z-e^{i\theta}|^2}$ is the *Poisson kernel*.

- Another approach is via random walks (Brownian motion). The formula is
  $$u(z) = \mathbb{E}[f(p_T(z))],$$
  where $p_t(z)$ is a random (Brownian motion) path starting at z, T is the time when it first hits the unit circle, and $\mathbb{E}[\cdot]$ denotes the expected value over all paths. This u satisfies the mean value property, hence is harmonic, and it agrees with f on S^1.

Dirichlet problem in other domains. One can use the above solve the Dirichlet problem for any simply connected domain bounded by a simple closed curve. First conformally map to the disc via a map g, solve the problem there, and then transfer back (use that $u \circ g$ is harmonic if u harmonic and g holomorphic).

Fluid flow. Fluid flow (incompressible, irrotational) around obstacle. Solve the problem first on $\mathbb{H}$ (flow lines are parallel to boundary), then transport conformally to other domains.