Homework 1 : MATH 4530

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Blackboard. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. Determine whether each of the following statements is true for all sets A, B, C, D:
   (a) $A - (A - B) = B$
   (b) $A - (B - A) = A - B$
   (c) $A \cap (B - C) = (A \cap B) - (A \cap C)$
   (d) $A \subset C$ and $B \subset D \Rightarrow A \times B \subset C \times D$.

2. Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0, 1$. Show that f^{-1} preserves inclusions, unions, intersections, and differences:
   (a) $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$.
   (b) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.
   (c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.
   (d) $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$.

   Show that f preserves inclusions and unions only:
   (e) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$.
   (f) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.
   (g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; and show $\subset$ cannot be replaced by $=$ in general.
   (h) $f(A_0 - A_1) \supset f(A_0) - f(A_1)$; and show $\supset$ cannot be replaced by $=$ in general.

3. (a) A real number x is said to be *algebraic* (over the rationals) if it satisfies some polynomial equation of positive degree

   $$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

   with rational coefficients a_i. Show that the set of algebraic numbers is countable.
   (b) A real number is said to be *transcendental* if it is not algebraic. Assuming the reals are uncountable, show that the transcendental numbers are uncountable. (It is a somewhat surprising fact that only a few transcendental numbers, for instance e and π, are familiar to us. Even proving these two numbers transcendental is highly nontrivial.)

4. Let $\mathcal{A}$ be some collection of disjoint open intervals in $\mathbb{R}$. Is $\mathcal{A}$ countable?