Homework 4: MATH 4530

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Blackboard. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. Let $\mathcal{T}$ be the collection of all subsets of $\mathbb{R}^n$ of the form

$$\mathbb{R}^n - \bigcap_{f \in P} \{x : f(x) = 0\},$$

where P is some collection of polynomials in n variables. Show that $\mathcal{T}$ defines a topology. (This is known as the *Zariski Topology*; it is of central importance in algebraic geometry.) Show that for each $n \geq 1$, it is not Hausdorff. Is it finer, coarser, or incomparable to the standard topology on $\mathbb{R}^n$?

2. Suppose X_α are sets, for $\alpha \in I$, where I is some index set. Recall the *disjoint union*

$$D = \coprod_{\alpha \in I} X_\alpha := \bigcup_{\alpha \in I} \{(x, \alpha) : x \in X_\alpha\}.$$

For each α, there is a canonical injection $X_\alpha \to D$ given by $x \mapsto (x, \alpha)$.

Now suppose each X_α is equipped with a topology. We equip D with the finest topology so that all the canonical injections are continuous. Give a simple characterization of openness of a subset of D in this topology.

3. Let S^1 be the circle with the standard topology. Consider the disjoint union space

$$E := \left(\coprod_{i \in \mathbb{Z}} S^1 \right) \sqcup \left(\coprod_{j \in \mathbb{Z}} [0, 1) \right).$$

Show that there exists a continuous bijection $E \to E$ which is *not* a homeomorphism.

4. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

5. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise.} \end{cases}$$

Show that F is continuous in each variable separately, but that F is not continuous.

6. Let S be the subset of $\mathbb{R}^{\mathbb{Z}_+}$ consisting of those sequences that are eventually zero, i.e. all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of S in $\mathbb{R}^{\mathbb{Z}_+}$ in the box and product topologies?