Homework 6: MATH 4530

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Blackboard. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. (a) Define an equivalence relation on $\mathbb{R}^2$ by

$$(x_0, y_0) \sim (x_1, y_1) \quad \text{if} \;\; x_0 + y_0^2 = x_1 + y_1^2.$$

   Let $\mathbb{R}^2/\sim$ be the quotient. Show that it is homeomorphic to a familiar space. (Hint: consider $g(x, y) = x + y^2$)

   (b) Repeat the previous part for the equivalence relation

$$(x_0, y_0) \sim (x_1, y_1) \quad \text{if} \;\; 3x_0 + 5y_0^2 = 3x_1 + 5y_1^2.$$

2. (a) Show that no two of the spaces $(0, 1), (0, 1], [0, 1]$ are homeomorphic. (Hint: think about what happens if a point is removed from each of these)

   (b) Show that $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

3. (a) Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.

   (b) Let $f : [0, 1] \to [0, 1]$ be continuous. Show that there exists a point $x \in [0, 1]$ with $f(x) = x$. (This is a prototypical example of a *fixed point theorem*)

4. Show that if U is an open connected subspace of $\mathbb{R}^2$ then U is path connected. (Hint: Show that given $x \in U$, the set of points that be joined to it by a path within U is both open and closed).