Homework 9: MATH 4530

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Blackboard. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

Notation: We denote by $[X, Y]$ the set of homotopy classes of continuous maps from X to Y.

1. Given a space X, let $\pi_0(X)$ denote the set of path components of X.
   (a) Show that there is a bijection between $\pi_0(X)$ and the set $[\{0\}, X]$ (here $\{0\}$ is the one point space, on which there is a unique topology).
   (b) Let $f : X \to Y$ be a continuous map. Show that this induces a map $f_* : \pi_0(X) \to \pi_0(Y)$. Show that if $g : X \to Y$ is homotopic to f, then $f_* = g_*$.
2. Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.
3. A space X is *contractible* if the identity map $i : X \to X$ is nullhomotopic (i.e. homotopic to a constant map).
   (a) Show that $[0, 1]$ and $\mathbb{R}$ are contractible.
   (b) Show that if X is contractible, then it is path connected.
   (c) Show that if X is contractible, then it is simply connected.
   (d) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.
   (e) Show that if X is contractible and Y path connected, then $[X, Y]$ has single element.
4. A subset $A \subset \mathbb{R}^n$ is *star-shaped* if for some point $a_0 \in A$, every line segment joining a_0 to another point in A lies in A.
   (a) Find a star-shaped set that it is not convex.
   (b) Show that if A is star-shaped, then A is contractible.
5. Given $x_0, x_1 \in X$, and α a path homotopy class of paths from x_0 to x_1, let $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ be the homomorphism $\gamma \mapsto \alpha^{-1} * \gamma * \alpha$. Now suppose that X is path connected, and for every α, β path homotopy classes of paths from x_0 to x_1, we $\hat{\alpha} = \hat{\beta}$. Show that $\pi_1(X, x_0)$ is abelian (i.e. $\gamma_1 * \gamma_2 = \gamma_2 * \gamma_1$ for all $\gamma_1, \gamma_2 \in \pi_1(X, x_0)$).