Homework 10: MATH 4530

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Blackboard. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. Let X and Y be topological spaces. Prove that if Y is contractible, then any two continuous maps $f, g : X \to Y$ are homotopic.

2. Let Y be a space with the discrete topology. Show if that if $p : X \times Y \to X$ is projection on the first coordinate, then p is a covering map.

3. Let $\pi : E \to B$ be a covering map, with B connected. Show if the fiber $\pi^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $\pi^{-1}(b)$ has exactly k elements for every $b \in B$. (This k is the *degree* of the cover.)

4. Let $q : X \to Y$ and $r : Y \to Z$ be covering maps. Show that if $r^{-1}(z)$ is finite for each $z \in Z$, then $r \circ q$ is a covering map.

   Challenge problem (not for credit): show that the statement is false without the condition on finiteness of fibers.

5. Let $\pi : \mathbb{R} \to S^1$ be given by $\pi(x) = (\cos 2\pi x, \sin 2\pi x)$, and consider the product map $\pi \times \pi : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$. Consider the path

   $$\gamma(t) = ((\cos 2\pi t, \sin 2\pi t), (\cos 4\pi t, \sin 4\pi t))$$

   in $S^1 \times S^1$. Sketch what γ looks like on the torus. Find a lift of $\tilde{\gamma}$ of γ to a path in $\mathbb{R} \times \mathbb{R}$ and sketch it.

6. Let $\pi : E \to B$ be a covering map, with E path connected. Show that if B is simply connected, then π is a homeomorphism.