Homework 1: MATH 6120

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Gradescope. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. Let v, w be distinct points in $\mathbb{C}$, and $\rho > 0$. Show that the locus of points $z \in \mathbb{C}$ that satisfy

$$|z - v| = \rho|z - w|$$

   is a circle or a line.

2. Show that it's impossible to define a *total ordering* on $\mathbb{C}$, i.e. a relation $\succ$ beween complex numbers so that:

   (a) For any $z, w \in \mathbb{C}$, exactly one of the following holds: $z \succ w$, $w \succ z$, or $z = w$.

   (b) For all $z_1, z_2, w \in \mathbb{C}$ the relation $z_1 \succ z_2$ implies $z_1 + w \succ z_2 + w$.

   (c) For all $z_1, z_2, w \in \mathbb{C}$ with $w \succ 0$, the relation $z_1 \succ z_2$ implies $z_1 w \succ z_2 w$.

3. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

   Use these to show that the logarithm function defined by

$$\log z = \log r + i\theta \;\; \text{where} \;\; z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

   is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

4. Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(x + iy) = \sqrt{|x||y|}$ for all $x, y \in \mathbb{R}$. Show that f satisfies the Cauchy-Riemann equations at 0, but f is not holomorphic at 0.

5. Let $U \subset \mathbb{C}$ open, and $f : U \to \mathbb{C}$ holomorphic. Show that if $\text{Re}(f)$ is constant, then f is constant.