Homework 2 : MATH 6120

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Gradescope. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. We define the exponential function $\exp : \mathbb{C} \to \mathbb{C}$ via the power series $\sum_{n \geq 0} \frac{z^n}{n!}$. Show that $\exp(z+w) = \exp(z)\exp(w)$ for all $z, w \in \mathbb{C}$.

2. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x^2} & \text{if } x > 0. \end{cases}$$

   Prove that f is infinitely differentiable on $\mathbb{R}$, but f does not have a convergent power series expansion $\sum_n a_n x^n$ near the origin.

3. (a) For each of the power series $\sum_n nz^n$ and $\sum_n z^n/n^2$, find all values $z \in \mathbb{C}$ for which the series converges.

   (b) Find $z_1, z_2 \in \mathbb{C}$ with $|z_i| = 1$ such that the series $\sum z^n/n$ converges and diverges at z_1, z_2, respectively.

4. Find the power series expansion of $\log z$ centered at $z = 1$.

5. Find a power series $f(z) = \sum_n a_n z^n$ that converges absolutely on the unit disc, but does not extend to a holomorphic map on any larger domain. (That is, if g is holomorphic on an open set U containing the unit disk Δ and $f(z) = g(z)$ on Δ, then $U = \Delta$.)

6. Find all values $z \in \mathbb{C}$ at which $\exp : \mathbb{C} \to \mathbb{C}$ is conformal.