Homework 3 : MATH 6120

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Gradescope. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. Show (without using Cauchy's theorem or any of its consequences) that if $|a| < r < |b|$, then

$$\int_{S_r} \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b},$$

where S_r denotes the circle of radius 1 centered at the origin.

2. Show that

$$\int_0^\infty \cos(x^2) dx = \frac{\sqrt{2\pi}}{4}.$$

(See Stein-Shakarchi Ch.2 Exercise 1 for hints).

3. Suppose $U \subset \mathbb{C}$ open, and let $f_1, f_2, \ldots : U \to \mathbb{C}$ be continuous functions. Let $\gamma \subset U$ be a piecewise smooth curve, and suppose the series $\sum_{n=1}^\infty f_n(z)$ converges uniformly on γ to some function f. Show that the series can be integrated term by term:

$$\int_\gamma f(z) dz = \sum_{n=1}^\infty \int_\gamma f_n(z) dz.$$

4. Let $U \subset \mathbb{C}$ be open, and let R be a rectangle, such that R and its interior are contained in U. Let z_0 be a point in the interior of U. Suppose that $f : U - z_0 \to \mathbb{C}$ is holomorphic and *bounded*. Show that

$$\int_R f(z) dz = 0.$$

5. The Weierstrass approximation theorem states that every continuous function on the closed interval $[0, 1]$ can be uniformly approximated arbitrarily well by polynomials. Can every continuous function on the closed unit disk in $\mathbb{C}$ be approximated arbitrarily well by polynomials in z?