Homework 4 : MATH 6120

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Gradescope. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. Recall we defined $\sin, \cos$ for complex arguments using their standard power series. Give formulas for the real and imaginary parts of $\sin(x+iy)$ and $\cos(x+iy)$ in terms of standard real-variable elementary functions. Find the set of zeros of sin and cos as functions from $\mathbb{C} \to \mathbb{C}$.

2. Suppose f is an entire function such that for any each $z_0 \in \mathbb{C}$, at least one coefficient in the power series expansion
$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$
is zero. Prove that f is a polynomial.

3. Let $U \subset \mathbb{C}$ be open and connected. Let $f_1, \ldots, f_n : U \to \mathbb{C}$ be holomorphic functions. Suppose $\sum_i |f_i(z)|$ is constant on U. Show that each f_i is constant on U.

4. Let $u : \Delta \to \mathbb{R}$ be a real-valued harmonic function on the unit disk. Show that there exists a holomorphic function $f : \Delta \to \mathbb{C}$ with $\mathrm{Re}(f) = u$. To what extent is such an f unique?