

## Homework 5 : MATH 6120

**Collaboration Policy :** You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

**Submission:** Upload a .pdf file using the page for this assignment in Gradescope. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. Find an example of an open set  $U \subset \mathbb{C}$  and a harmonic function  $u : U \rightarrow \mathbb{R}$  such that there does *not* exist a holomorphic function  $f : U \rightarrow \mathbb{C}$  with  $\operatorname{Re}(f) = u$ .
2. Find all rational functions  $f(z)$  that preserve the unit circle  $S^1$ .
3. Prove that for any polynomial  $p(z)$ , there exists a  $z \in S^1$  with  $|\bar{z} - p(z)| \geq 1$ .
4. Prove that there is a sequence of polynomials  $\{p_k(z)\}$  that converge pointwise on  $\mathbb{C}$  to the following function:

$$f(z) = \begin{cases} -1 & \text{if } \operatorname{Re}(z) < 0, \\ 0 & \text{if } \operatorname{Re}(z) = 0, \\ e^z & \text{if } \operatorname{Re}(z) > 0. \end{cases}$$