Homework 7 : MATH 6120

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Gradescope. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. Suppose $f : \Delta^* \to \mathbb{C}$ is holomorphic, and
$$|f(z)| \leq 1/\sqrt{|z|}$$
for all $z \in \Delta$. Show that f extends to a holomorphic function $\Delta \to \mathbb{C}$.

2. Show that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and *injective*, then $f(z) = az + b$ for some $a, b \in \mathbb{C}$. (Hint: consider the behavior at 0 of $f(1/z)$).

3. Find the residue of $\sin^3(1/z)$ at 0.

4. (a) Let $U \subset \mathbb{C}$ open, and let $f, g : U \to \mathbb{C}$ holomorphic. Show that
$$Res(gf'/f, p) = g(p)Mult_p(f)$$
(where $Res(\cdot, p)$ is the residue at p, and the left hand side is the multiplicity of vanishing of f at p). What does this give for $g(z) = z$ when we apply the residue formula?

   (b) Let
$$p_u(z) = z^n + a_{n-1}(u)z^{n-1} + \cdots + a_0(u)$$
be a holomorphic *family* of polynomials, where each $a_i : \Delta \to \mathbb{C}$ is a holomorphic function. Suppose $p_0(z)$ has all roots simple (multiplicity 1). Show that there is some disc D containing 0 and holomorphic functions $r_i : D \to \mathbb{C}$ such that
$$p_u(z) = (z - r_1(u)) \cdots (z - r_n(u))$$
for all $u \in D$ and $z \in \Delta$.