Homework 8: MATH 6120

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Gradescope. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. Find all holomorphic functions $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

2. Let
$$f_n(z) = \sum_{k=0}^{n} z^k/k!.$$
Fix $R > 0$. Prove that for n large enough (depending on R), f_n has no zeros in $D(0, R)$.

3. Let G be the group of Mobius transformations. Show that any $M \in G$ is conjugate in G to either $f(z) = \lambda z$, for some $\lambda \in \mathbb{C}^*$, or to $f(z) = z + 1$.

4. Let $f(z) = \frac{az+}{cz+d}$ be a Mobius transformation. Show that the number of rational maps $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that
$$g(g(g(z))) = f(z)$$
for all z is either $1, 3$, or ∞. (Hint: use previous problem)

5. Construct an explicit sequence $z_n \to 0$ such that $\exp(1/z_n) \to 0$.

6. Prove or disprove: if $f : \Delta \to \mathbb{C}$ is holomorphic with n zeros, then f' has at least $n - 1$ zeros in Δ. Do the same, with "at least" replaced by "at most".