Homework 9: MATH 6120

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Gradescope. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

Several of the problems below concern the *hyperbolic metric* on $\mathbb{H}$, which is the Riemannian metric given by $\frac{1}{y^2}(dx^2 + dy^2)$. In other words, the length of a tangent vector at $x + iy$ is the length in the standard Euclidean metric multiplied by y.

1. Show that any automorphism of $\mathbb{H}$ preserves the hyperbolic metric (that is, it's an *isometry* for the metric). Are these the only isometries?

2. Show that any vertical line in $\mathbb{H}$ is a geodesic for the hyperbolic metric (i.e. for any pair of points p, q on the line, the shortest path between the points is along the line). Show that any semicircle in $\mathbb{H}$ that meets the boundary $\mathbb{R}$ at right angles is a geodesic.

3. Find a metric on Δ that is invariant under any automorphism of Δ.

4. Let $f : \Delta \to \Delta$ be holomorphic. Determine the possibilities for the set of fixed points of f.

5. Let $U \subset \mathbb{C}$ be open and let $\mathcal{F}$ be a family of real differentiable functions $U \to \mathbb{C}$. Suppose there exists a real number M such that all four partials of f at every point are bounded in absolute value by M. Show $\mathcal{F}$ is equicontinuous. Does the reverse implication hold, i.e. does equicontinuity imply boundedness of partials?

6. Show that $f(z) = -\frac{1}{2}(z + 1/z)$ gives a biholomorphism from the upper half disc to the upper half plane.