Homework 10: MATH 6120

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Gradescope. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. Let $U \subset \mathbb{C}$ be open and connected. Let $f_n \to f$ be holomorphic functions, where the convergence is uniform on compact subsets. Suppose f is not identically zero. Prove that if $f(w) = 0$, then we can write $w = \lim z_n$, where $f_n(z_n) = 0$ for all n sufficiently large. Does this hold if we assume only that U is open (but not that it is connected)?

2. Determine the automorphism group of $\mathbb{C} - \{0, 1\}$.

3. Show that
$$z \mapsto \int_0^z \zeta^{-\beta_1}(1-\zeta)^{-\beta_2} d\zeta,$$
with $0 < \beta_1, \beta_2 < 1$ and $1 < \beta_1 + \beta_2 < 2$, maps $\mathbb{R} \cup \infty$ to a triangle. What happens when $\beta_1 + \beta_2$ is 1, and what happens when it is < 1?

4. Find an example of a holomorphic map $F : \mathbb{H} \to \mathbb{C}$ that extends to a continuous map on $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ that maps $\mathbb{R} \cup \infty$ bijectively to the boundary of a *non-convex* polygon.

5. Find a biholomorphism $f : \Delta \to \mathbb{C} - [1, \infty)$ with $f(0) = 0$ and $f'(0) > 0$.

6. Let $\mathcal{S} \subset \mathbb{C}$ be the interior of a square. Find an explicit example of a diffeomorphism $F : \Delta \to \mathcal{S}$ that does not have a continuous extension $\overline{\Delta} \to \overline{\mathcal{S}}$.