Homework 11: MATH 6120

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Gradescope. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

1. A *circular polygon* $\mathcal{C}$ is a simply connected open set of $\mathbb{C}$ whose boundary is a union of circular arcs and is homeomorphic to a circle. Show that any Riemann map $\Delta \to \mathcal{C}$ extends to a homeomorphism $\overline{\Delta} \to \overline{\mathcal{C}}$.

2. Let $f : \Delta \to \Delta$ be a holomorphic function that extends continuously to $\overline{\Delta} \to \overline{\Delta}$, with $f(S^1) \subset S^1$. Show that f can be extended to a holomorphic function on $\mathbb{C}$ with finitely many points removed.

3. Let $U := \{x + iy : y \geq x^2\} \subset \mathbb{C}$, and consider ∂U its boundary in $\mathbb{C}$. Let $f : U \to \mathbb{C}$ be a holomorphic function that extends continuously to $F : U \cap \partial U \to \mathbb{C}$ with $F(\partial U) \subset \partial U$. Show that f can be extended to a holomorphic function on a strictly larger open set $V \supsetneq U$.

4. Find two bounded open subsets of $\mathbb{C}$ that are homeomorphic but not biholomorphic.

5. Interpret the function $\log z$ on $\mathbb{H}$ as a degenerate Schwarz-Christoffel map with image a degenerate polygon.