Homework 12: MATH 6120

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

Submission: Upload a `.pdf` file using the page for this assignment in Gradescope. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly.

See Stein-Shakarchi Ch. 9 Exercises for hints.

1. Suppose that $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is a meromorphic function with two periods ω_1, ω_2, such that the ratio ω_1/ω_2 is real.
   (a) Suppose ω_1/ω_2 is rational. Show that the periodicity assumption is equivalent to being periodic with respect to some single ω'.
   (b) If ω_1/ω_2 is irrational, show f is constant.

2. Given a lattice $\Lambda \subset \mathbb{C}$, addition of complex numbers naturally descends to a group law on the torus $\mathbb{C}/\Lambda$. Let $F : \mathbb{C}/\Lambda \to \widehat{\mathbb{C}}$ be a meromorphic function with zeros at $z_1, \ldots, z_n$ and poles at $p_1, \ldots, p_m$ (appearing with multiplicity equal to the order). Prove that
$$\sum z_i - \sum p_j = 0,$$
   where the addition is with respect to the group law described above.

3. Given a lattice $\Lambda \subset \mathbb{C}$, show that the series
$$\sum_{\omega \in \Lambda^*} \frac{1}{|\omega|^2}$$
   is divergent.

4. Let S be the strip $\{z : \mathrm{Im}(z) > 0, -\pi/2 < \mathrm{Re}(z) < \pi/2\}$, and let $f : \mathbb{H} \to S$ be the Riemann map whose extension to the boundary of $\mathbb{H}$ takes $(-1, 1, \infty)$ to $(-\pi/2, \pi/2, \infty)$. Show that f^{-1} can be extended to a map $\mathbb{C} \to \mathbb{C}$, and prove that it coincides with a familiar function.