Homework 1: MATH 6210

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

1. Show that a countable union of countable sets is countable (Hint: think about the proof that $\mathbb{Q}$ is countable).

2. Let $\mathcal{C} \subset [0,1]$ be the standard middle-thirds Cantor set. Let $\chi_{\mathcal{C}} : \mathbb{R} \to \mathbb{R}$ be the indicator function of $\mathcal{C}$. Prove that $\chi_{\mathcal{C}}$ is *Riemann* integrable, and compute the value of the integral. (We briefly went over the definition of Riemann integrability in terms of existence of limits of Riemann sums, but you may need to review it more thoroughly).

3. Let $\mathcal{C} \subset [0,1]$ be the standard middle-thirds Cantor set, as constructed in class. Show that $\mathcal{C}$ admits the following equivalent description: it is the set of points that can be written as

$$\sum_{k=1}^{\infty} a_k \cdot 3^{-k},$$

   where a_k are all either 0 or 2 (in other words, the points with a ternary expansion that have no 1's in them). Use this to show that $\mathcal{C}$ is an uncountable set.

4. Show that every open subset of $\mathbb{R}$ is the countable union of a collection of disjoint open intervals.

5. Let $S = \mathbb{Q} \cap [0,1]$. Show that if $I_1, \ldots, I_N$ is a finite collection of intervals covering S, then $\sum_{j=1}^{N} |I_j| \geq 1$.