# Homework 2: MATH 6210

**Collaboration Policy** : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

*Note:* For problems 1 and 2, use the following definition (which we will discuss in class on Monday). A set $E \subset \mathbb{R}$ is *measurable* if for all $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}$ with $E \subset U$ and $\mu^*(U - E) < \epsilon$.

1. Prove that a countable intersection of measurable sets is measurable.

2. Prove that each of the following sets is measurable, and compute its measure:

   (a) The (standard middle-thirds) Cantor set $\mathcal{C}$.

   (b) The set $[0,1] - \mathbb{Q}$ of irrational numbers in the unit interval.

3. Let $E_1, E_2, \ldots$ be subsets of $\mathbb{R}$ and $\delta > 0$ such that $d(E_i, E_j) > \delta$ for all $i, j$ with $i \neq j$. Prove that

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu^*(E_i).$$

4. We define variants of the standard Cantor sets, called *thickened Cantor sets*, as follows. The construction depends on a sequence of positive real numbers $\ell_1, \ell_2, \ldots$ Take $\mathcal{C}'_0 = [0,1]$, and construct $\mathcal{C}'_k$ inductively to be a union of disjoint closed intervals as follows. To form $\mathcal{C}'_k$ with $k \geq 1$ start with $\mathcal{C}'_{k-1}$ and remove from each of its closed intervals a centrally situated open interval of length $\ell_k$. Then define $\mathcal{C}' = \cap_{i=1}^{\infty} \mathcal{C}'_k$. Note that the standard Cantor set arises by taking $\ell_1 = 1/3, \ell_2 = 1/9, \ldots$.

   (a) Prove that the $\ell_i$ can be chosen so that $\mu^*(\mathcal{C}') > 0$.

   (b) Prove that for any choice of $\ell_i$, $\mathcal{C}'$ is *nowhere dense*, i.e. every interval $(a,b) \subset [0,1]$ contains a subinterval disjoint from $\mathcal{C}'$.

5. Given a set $E \subset \mathbb{R}$, let

$$U_n^E := \{x \in \mathbb{R} : d(x, E) < 1/n\}.$$

   Find an open, bounded set $E$ such that $\mu^*(E) \neq \lim_{n\to\infty} \mu^*(U_n^E)$. (Hint: use previous problem)