Homework 4 : MATH 6210

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

1. Let $E_1, E_2, \dots$ be measurable subsets of $\mathbb{R}$ with $\mu(E_i) = \infty$ for all i. Must $\bigcap_{i=1}^{\infty} E_i$ also have infinite measure?

2. (Corrected) We say that a subset of $\mathbb{R}^d$ is a G_δ *set* if it can be expressed as a countable intersection of open sets. Let $E \subset \mathbb{R}^d$. Show that E is measurable if and only if E can be written as $G - Z$ where G is a G_δ set and Z is a set with $\mu(Z) = 0$.

3. Show that if $E \subset \mathbb{R}^d$ is measurable then
$$\mu(E) = \sup_{F \text{ closed, } F \subset E} \mu(F).$$

4. Prove that there is some positive integer n for which there exist pairwise disjoint sets $E_1, \dots, E_n$ in $\mathbb{R}$ with
$$\mu^*(E_1 \cup \cdots \cup E_n) \neq \mu^*(E_1) + \cdots + \mu^*(E_n).$$
Show that in fact one can take $n = 2$. (Hint: think about outer measure of the Vitali sets we constructed in class).

Extra-credit (worth half a normal problem): If $E \subset \mathbb{R}$ is measurable, can E can be written as $G \cup Z$ where G is a G_δ set and Z is a set with $\mu(Z) = 0$?