Homework 5 : MATH 6210

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

1. Find a set that is not Lebesgue measurable and contains an uncountable subset of Lebesgue measure 0.

2. Let X be any set. Let $\{\Sigma_\alpha\}_{\alpha\in A}$ be an arbitrary collection of σ-algebras on X. Show that the intersection

$$\bigcap_{\alpha\in A}\Sigma_\alpha$$

is also a σ-algebra on X. Is the union $\bigcup_{\alpha\in A}\Sigma_\alpha$ necessarily a σ-algebra?

3. Let Σ be the collection of subsets $S\subset\mathbb{R}$ such that either (i) S has Lebesgue measure 0, or (ii) $\mathbb{R}-S$ has Lebesgue measure 0. Show that Σ is a σ-algebra on $\mathbb{R}$.

4. Show that any Lebesgue measurable set $X\subset\mathbb{R}$ is a union of a Borel set and a set of Lebesgue measure 0.

5. Let X be an uncountable set, and $\Sigma=\mathcal{P}(X)$ be the σ-algebra of all subsets of X. Consider the following function $m:\Sigma\to\mathbb{R}$ whose value on a set S is given by

$$m(S)=\begin{cases}1 \text{ if } X-S \text{ is countable}\\ 0 \text{ otherwise}\end{cases}$$

Is m a measure on the measurable space (X,Σ)?