Homework 06: MATH 6210

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

1. Consider the *Cantor dust* fractal $\mathcal{D} \subset \mathbb{R}^2$ defined as follows. Let $\mathcal{D}_0$ be the unit square, and construct sets that are unions of disjoint squares inductively as follows. To form $\mathcal{D}_n$, start with $\mathcal{D}_{n-1}$, break each of its squares into 9 smaller squares, and remove all but the four squares at the corners. We then define $\mathcal{D} = \cap_{j=0}^{\infty} \mathcal{D}_j$.

   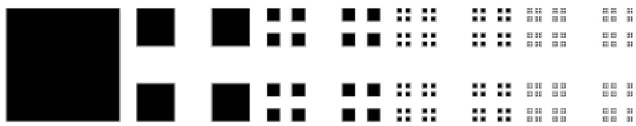

   Figure 1: Constructing the Cantor dust. Stages $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ are shown

   Compute heuristically the dimension of $\mathcal{D}$ using self-similarity, as we did in class for the Cantor set. (This does not need to be justified rigorously).

2. Prove the following fact stated in class. If d is a non-negative real number, and $E_1, E_2 \subset \mathbb{R}^n$ are positively separated, i.e. $d(E_1, E_2) > 0$, then $h_d^*(E_1 \cup E_2) = h_d^*(E_1) + h_d^*(E_2)$.

3. Does there exist some positive integer n, non-negative real $d < n$, and countably many sets $E_1, E_2, \ldots \subset \mathbb{R}^n$ such that $h_d(E_i) < \infty$ for all i, and $\bigcup_{i=1}^{\infty} E_i = \mathbb{R}^n$.

4. Prove (rigorously) that the Hausdorff dimension α of the standard Cantor satisfies $\alpha \leq \log_3 2$.

5. Compute the Hausdorff dimension of the set $\{0, 1, 1/2, 1/3, 1/4, \cdots\} \subset \mathbb{R}$.