# Homework 8: MATH 6210

**Collaboration Policy** : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

1. Prove monotonicity of the integral: if $f, g : \mathbb{R} \to [-\infty, +\infty]$ are measurable with $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, then $\int f \leq \int g$.

2. Let $a_1, a_2, \ldots \in \mathbb{R}_{\geq 0}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} a_n & \text{if } x \in (n, n+1) \\ 0 & \text{else.} \end{cases}$$

   Prove that $f$ is measurable and rigorously compute the Lebesgue integral $\int f$.

3. Is it true that for any sequence of measurable functions $f_i : \mathbb{R} \to \mathbb{R}_{\geq 0}$ with $f_1 \geq f_2 \geq \cdots$ that

$$\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n?$$

   (I.e. does a version of the monotone convergence theorem hold for *decreasing* functions?)

4. Prove Chebyshev's inequality: if $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a measurable function with $\int f < \infty$ and $\alpha > 0$, then

$$\mu\left(\{x : f(x) > \alpha\}\right) \leq \frac{1}{\alpha} \int f.$$

5. Let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be measurable. Prove that $\int f = 0$ if and only if $f$ equals 0 almost everywhere (i.e. $\mu(\{x : f(x) \neq 0\}) = 0$).

*Optional problem not for credit:* Is there a function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ that is *Riemann* integrable and the set of points where $f$ is discontinuous is dense in $\mathbb{R}$.