Homework 10: MATH 6210

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

1. Let I be a finite length interval in $\mathbb{R}$. Show that for every $\epsilon > 0$, there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ with $\|f - \chi_I\|_{L^1} < \epsilon$.

2. Let $C \subset \mathbb{R}$ be a closed set, and let $f : C \to \mathbb{R}$ be a continuous function. Show that there is a continuous function $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = f(x)$ for all $x \in C$.

3. Suppose that $f, f_1, f_2, \ldots \in L^1(\mathbb{R})$ and $f_n \to f$ in $L^1(\mathbb{R})$. Prove that there exists a subsequence $\{f_{n_k}\}$ and a function $g \in L^1(\mathbb{R})$ such that for all k, $|f_{n_k}| \leq g$ almost everywhere.

4. Show that there exists an integrable *continuous* function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\limsup_{x \to \infty} f(x) = +\infty.$$

Optional problem: Show that the vector space of Riemann integrable functions $f : \mathbb{R} \to \mathbb{R}$ equipped with the norm $\|f\| := \int |f|$ is not complete.