Homework 11 : MATH 6210

Collaboration Policy : You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

1. Show that there exists a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that for any $g : \mathbb{R} \to \mathbb{R}$ *continuous*, we have that
$$\mu(\{x : g(x) \neq f(x)\}) > 0.$$
2. Suppose $1 \leq p < q < r < \infty$ and $f \in L^p(\mathbb{R}) \cap L^r(\mathbb{R})$. Must f also be in $L^q(\mathbb{R})$?
3. Show that there exists a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in L^p(\mathbb{R})$ for all p with $1 \leq p < \infty$, but $f \notin L^\infty(\mathbb{R})$.
4. Let p, q with $1 \leq p, q < \infty$. Suppose that $p \neq q$. Show that there exists an $f : \mathbb{R} \to \mathbb{R}$ measurable such that $f \in L^p(\mathbb{R})$ and $f \notin L^q(\mathbb{R})$.
5. Prove that if $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, and $g \in L^\infty(\mathbb{R})$, then the product fg is in $L^p(\mathbb{R})$ and
$$\|fg\|_p \leq \|f\|_p \|g\|_\infty.$$

Optional problem: Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function that vanishes outside of $[0, 1]$. Suppose that $f \in L^\infty(\mathbb{R})$. Show that $f \in L^p(\mathbb{R})$ for any p with $1 \leq p < \infty$. Then show that
$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$