Homework 1: MATH 6640

Instructions: Submit solutions to **5** of the following 6 problems.

Submission: Upload a `.pdf` file using the page for this assignment in Canvas. You may produce this either (i) electronically, or (ii) by hand, legibly, and then scanned, legibly. It is generally easy to convert a file from some other format, such as .docx or .jpeg, to .pdf.

Collaboration Policy You may, in fact are encouraged to, work on the problems with other students. You must write up your solutions by yourself.

1. Show that given a disk D in the Euclidean plane and a tangent vector v at a point in the disc, there exists a circle or line that is tangent to v and that meets the boundary of D at right angles.

2. Given a pair of geodesics in the hyperbolic plane, show that there is an isometry taking one to the other.

3. Let C be a circle in S^2 (thought of as $\mathbb{R}^2 \cup \{\infty\}$). Let $\phi : S^2 \to S^2$ be a map that (i) is the identity on C, (ii) preserves setwise any circle perpendicular to C, and (iii) exchanges interior and exterior of C. Show that ϕ must be inversion about C.

4. Prove that the orientation preserving isometry group of $\mathbb{H}$ is contained in $PSL_2(\mathbb{R})$. You may use the fact that for a connected Riemannian manifold, an isometry is determined by what it does to a frame, i.e. a basis for the tangent space at a point. Bonus: prove this fact.

5. Let C_1, C_2, C_3 be three distinct circles in the plane, and x_{12}, x_{13}, x_{23} be three distinct points. Suppose that C_i and C_j are tangent at x_{ij} for each i, j with $i \neq j$. Prove that there are exactly two circles that are tangent to all three of C_1, C_2, C_3.

6. Do Exercise 1.2.5 (a) in the Thurston/Levy book, proving that a particular mechanical linkage performs inversion. For fun, you might also try (b), a construction project.