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Equations of linear subvarieties of strata of differentials

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We investigate the closure $\overline{M}$ of a linear subvariety M of a stratum of meromorphic differentials in the multiscale compactification constructed by Bainbridge, Chen, Gendron, Grushevsky and Möller. Given the existence of a boundary point of M of a given combinatorial type, we deduce that certain periods of the differential are pairwise proportional on M, and deduce further explicit linear defining relations. These restrictions on linear defining equations of M allow us to rewrite them as explicit analytic equations in plumbing coordinates near the boundary, which turn out to be binomial. This in particular shows that locally near the boundary $\overline{M}$ is a toric variety, and allows us to prove existence of certain smoothings of boundary points and to construct a smooth compactification of the Hurwitz space of covers of $\mathbb{P}^1$. As applications of our techniques, we give a fundamentally new proof of a generalization of the cylinder deformation theorem of Wright to the case of real linear subvarieties of meromorphic strata.

37F34; 14H15, 32G15

1 Introduction

For an n–tuple of integers $\mu = (m_1, \dots, m_n)$ satisfying $m_1 + \dots + m_n = 2g - 2$, $\Omega\mathcal{M}_{g,n}(\mu)$ denotes the stratum of meromorphic differentials of type μ, that is, the locus of triples $(X, \underline{x}, \omega)$ where $\underline{x} = \{x_1, \dots, x_n\}$ is a collection of distinct marked

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points on a smooth genus g Riemann surface X and ω is a nonzero meromorphic differential on X such that its divisor of poles and zeroes is $\operatorname{div}(\omega) = \sum m_i x_i$. We call points $(X, \underline{x}, \omega) \in \Omega\mathcal{M}_{g,n}(\mu)$ *flat surfaces of type* μ, and will also write $\underline{z} \subset X$ for the set of all zeroes of ω, that is, all x_i such that $m_i \geq 0$, and $\underline{p} \subset X$ for the set of all poles of ω. A natural set of local coordinates on the stratum is given by period coordinates: the integrals of ω over a chosen basis of $H_1(X \setminus \underline{p}, \underline{z}; \mathbb{Z})$. The group $\mathrm{GL}^+(2,\mathbb{R})$ naturally acts on the stratum, linearly in period coordinates. This dynamical system is the central object of study in Teichmüller dynamics.

The foundational results of Eskin and Mirzakhani [11] and Eskin, Mirzakhani and Mohammadi [12] show that, for holomorphic strata (that is, if all m_i are positive), the closure, in the Euclidean topology, of any $\mathrm{GL}^+(2,\mathbb{R})$–orbit is cut out by linear equations with real coefficients. Throughout this paper, when we say "linear equation", we mean an equation with no constant term. Filip [13] showed that such orbit closures, commonly called *affine invariant manifolds*, are algebraic varieties.

In this paper, we study a more general class of subvarieties of the strata than affine invariant manifolds. An *algebraic* subvariety $M \subseteq \Omega\mathcal{M}_{g,n}(\mu)$ of complex codimension m is called a *linear* (*sub*)*variety* if at any point it is locally a finite union of linear subspaces, in period coordinates. Most of our results are in this generality, i.e. allowing complex coefficients and allowing meromorphic strata — note in particular that a linear equation with complex coefficients does not need to be preserved by the $\mathrm{GL}^+(2,\mathbb{R})$–action. While we require M to be algebraic, note that recent examples of Bakker and Mullane (personal communication) indicate that a locus given locally in a meromorphic stratum by complex-linear equations may not be algebraic. Our paper continues in the spirit of the works of the second author [10] and the first author [4], using and developing degeneration techniques for flat surfaces to prove various properties of linear subvarieties. We also obtain information about the geometry of defining equations. This could be used to understand (or rule out the existence of certain) linear subvarieties in general, while our more precise results for affine invariant manifolds could provide tools for classifying $\mathrm{GL}^+(2,\mathbb{R})$–orbit closures.

Bainbridge, Chen, Gendron, Grushevsky and Möller [3] constructed the moduli space of multiscale differentials $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, such that $\Omega\mathcal{M}_{g,n}(\mu) \subset \Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is open dense, and such that the quotient $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu) = \Xi\overline{\mathcal{M}}_{g,n}(\mu)/\mathbb{C}^*$, where $\mathbb{C}^*$ scales the differential, is compact. A key property of both $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is that they are *smooth* (as complex orbifolds) algebraic varieties, with normal crossing boundary. A multiscale differential is a stable Riemann surface X together with a map $\ell\colon V(\Gamma) \twoheadrightarrow \{0, -1, \dots, -L(\Gamma)\}$ from the set of vertices of the dual graph Γ of X, and together with a collection η of meromorphic differentials η_v on the irreducible components X_v of X satisfying certain compatibility conditions (additionally one needs an enhancement of the level graph and a prong-matching; see below). The boundary $\partial\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is stratified, with *open* strata D_Γ indexed by enhanced level graphs Γ. Up to finite covers, each such stratum is a subspace of a product of certain strata of differentials given by certain residue conditions, and, as such, it (more precisely, some cover of it — see Remark 3.2) also admits local period coordinates, see Costantini, Möller and Zachhuber [9, Section 4] for much more on the geometry of the strata, which we will also use below.

In [4], the first author used a detailed analysis of the degeneration behavior of period coordinates to prove that the intersection $\partial M_\Gamma := \partial M \cap D_\Gamma$ of the boundary $\partial M := \overline{M} \cap \partial\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ of the closure of any linear variety M with any boundary stratum is locally given by linear equations in period coordinates on that boundary stratum. Here we investigate geometric properties and stratifications of boundaries of linear varieties. While our results are described and obtained locally near the boundary of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, they provide global geometric information: given the existence of some degeneration, they restrict the defining equations of M.

Defining equations of linear subvarieties

We fix once and for all a boundary point $p_0 \in D_\Gamma$, and work throughout in a small neighborhood U of p_0 in $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, which we may need to shrink further finitely many times. Recall that the edges $e \in E(\Gamma)$ of the dual graph are called horizontal or vertical depending on whether they connect vertices of same or different levels for ℓ; we write $E(\Gamma) =: E^{\mathrm{hor}}(\Gamma) \sqcup E^{\mathrm{ver}}(\Gamma)$. For a vertical edge $e \in E^{\mathrm{ver}}(\Gamma)$, we denote by $\ell(e^\pm)$ the levels of its top and bottom vertices, respectively.

We will consider the defining equations of M at a point $p = (X, \omega) \in M \cap U$. In general, M is an immersed, and not embedded, submanifold of the stratum, and we always require p to be a point where M is locally embedded (and so M is smooth at p). We then call a *defining equation* (*of M at p*) a linear equation F satisfied locally on M near p. We think of F as an equation $F(X, \omega) = \int_\beta \omega = 0$ for some relative homology class $\beta \in H_1(X \setminus \underline{p}, \underline{z}; \mathbb{C})$. The vector space of such defining equations F of M at p is locally constant on M in a neighborhood of p.

Recall that corresponding to any node $e \in E(\Gamma)$ there is the pinching curve $\Lambda_e \subset X$. Denote by λ_e the homology class of Λ_e, called the *vanishing cycle*. We say that F

crosses e if the intersection number $\langle \beta, \lambda_e \rangle$ is nonzero. We denote by $E^{\mathrm{hor}}(F) \subseteq E^{\mathrm{hor}}(\Gamma)$ the set of all horizontal nodes crossed by F.

We now define the notion of two nodes e_1 and e_2 being M*–cross-related*. A particular case of being M–cross-related is when there exists a defining equation F of M at p crossing the two nodes, i.e. $e_1, e_2 \in E^{\mathrm{hor}}(F)$, while there is no defining equation F' crossing a nonempty proper subset of the nodes of F, i.e. $E^{\mathrm{hor}}(F') \subsetneq E^{\mathrm{hor}}(F)$ and $E^{\mathrm{hor}}(F') \neq \varnothing$. We define M*–cross-equivalence classes* to be the equivalence classes in $E^{\mathrm{hor}}(\Gamma)$ generated by this particular case, and call two nodes M–cross-related if they are in the same equivalence class (see Definition 3.5 for a more detailed discussion).

Our first result is that the periods over the vanishing cycles for any two M–cross-related nodes are proportional.

Theorem 1.1 (periods over horizontal vanishing cycles are proportional) *For any pair* $e_1, e_2 \in E^{\mathrm{hor}}(\Gamma)$ *of* M*–cross-related horizontal nodes, the integrals of* ω *over the corresponding vanishing cycles* λ_{e_1} *and* λ_{e_2} *are proportional on* M*. In particular, the nodes are at the same level.*

In the case when M is an affine invariant manifold, the above can be deduced from Wright's cylinder deformation theorem [21]. We give a fundamentally new proof of a generalization of the cylinder deformation theorem (see Theorem 1.9 below), and the above is one of the key tools we use. See Example 3.8 for a simple example of a set of M–cross-related nodes and how the above theorem applies.

For periods over the vanishing cycles for vertical nodes, we have:

Theorem 1.2 (relations among periods over vertical vanishing cycles) *For any defining equation* F *of* M *at* p *and for any level* $i \le \top(F)$*, let* $e_1, \dots, e_k \in E^{\mathrm{ver}}(\Gamma)$ *be all the vertical nodes crossed by* F *such that* $\ell(e_j^+) > i \ge \ell(e_j^-)$*, i.e. all vertical nodes that cross the level transition between levels* $i+1$ *and* i*. Then the set of periods of* ω *over vanishing cycles* λ_{e_j} *satisfy a linear relation on* M *near* p*.*

Here recall that cutting X along Λ_e for all $e \in E^{\mathrm{ver}}(\Gamma)$ decomposes X into the level subsurfaces $X = \bigcup_i X_{(i)}$. For a collection of paths $\beta' \subset X$ we call its top level $\top(\beta')$ the maximal i such that $\beta' \cap X_{(i)} \neq \varnothing$. The top level $\top(F)$ denotes then the top level of the homology class $[\beta]$, which is defined to be the minimum of $\top(\beta')$ over all collections of paths β' representing the class $[\beta]$. In Proposition 3.11 we will prove a more precise version of this theorem that gives the coefficients of such a relation.

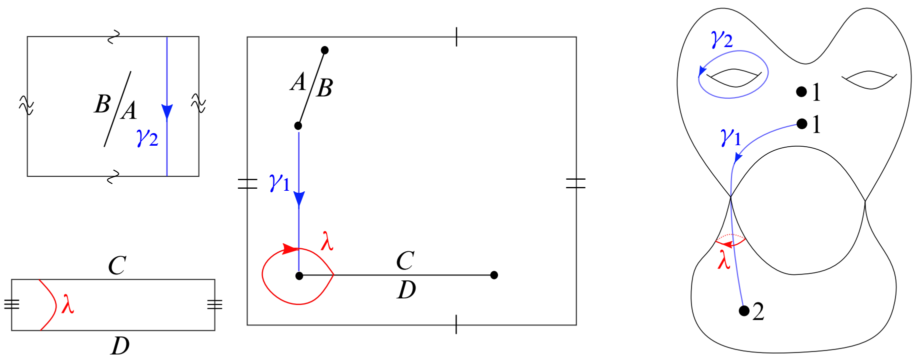

Figure 1: Left: a flat surface in $\Omega\mathcal{M}_{3,3}(1,1,2)$. Right: the limit as $\lambda \to 0$.

Example 1.3 Consider the flat surface in the stratum $\Omega\mathcal{M}_{3,3}(1,1,2)$ shown on the left in Figure 1. We claim that Theorem 1.2 implies that there is no linear subvariety M containing this surface that is locally described by the single equation

$$F = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega = 0.$$

In fact, if there were, we could degenerate, staying in M, by sending the period of the side labeled λ to 0 (and keeping the rest of the surface unchanged). This would give a family of surfaces converging to a point $p_0 \in \Xi\overline{\mathcal{M}}_{3,3}(1,1,2)$ with two levels and no horizontal nodes, in which the class λ is the vanishing cycle of a vertical node, with the curve Λ representing the homology class λ depicted in Figure 1, right. The period over this vanishing cycle is nonzero on all flat surfaces near p_0. The equation F crosses the (vertical) vanishing cycle λ, and no other vanishing cycles. By Theorem 1.2, there is a nontrivial linear relation among the periods of the vanishing cycles crossed by F on flat surfaces near p_0. Since this set of vanishing cycles is just $\{\lambda\}$, it follows that $\int_\lambda \omega = 0$ for all surfaces in M near p_0, which is impossible.

(In this particular case, such a linear manifold can also be ruled out using the cylinder deformation theorem [21], but in more complicated examples this would not be possible.) ◁

We further show that defining equations split into those that do not cross any horizontal nodes, and those that only cross horizontal nodes at their top level.

Theorem 1.4 (decomposition of linear equations) *Any defining equation F of M at p can be written as a sum*

(1-1) $$F = H_1 + \cdots + H_k + G$$

of defining equations of M *at* p (*possibly with* $k=0$) *such that*:

(1) *Each* H_j *crosses a primitive collection of horizontal nodes of level* $\top(H_j)$, *and no other horizontal nodes.*

(2) $E^{\mathrm{hor}}(H_j) \subseteq E^{\mathrm{hor}}(F)$ *for any* j.

(3) G *does not cross any horizontal nodes*: $E^{\mathrm{hor}}(G) = \varnothing$.

Here primitive means that there does not exist a defining equation H' of M at p such that $\varnothing \neq E^{\mathrm{hor}}(H') \subsetneq E^{\mathrm{hor}}(H_j)$. This theorem gives a restriction for the form of the defining equations, given the existence of a boundary point of M with enhanced level graph Γ.

Our methods allow us to control the dimensions of ∂M_Γ and in particular describe the boundary strata that may contain irreducible components of ∂M. Recall that the codimension $\operatorname{codim}_{\Xi\overline{\mathcal{M}}_{g,n}(\mu)} D_\Gamma$ of a boundary stratum is equal to $H(\Gamma)+L(\Gamma)$, where H is the number of horizontal edges and L is the number of levels below zero.

Theorem 1.5 (boundary components of M) *The general point of any irreducible component of the boundary* ∂M *is contained in an open boundary stratum* D_Γ *such that either* $L(\Gamma)=1$ *and* $H(\Gamma)=0$, *or* $L(\Gamma)=0$.

In the latter case, *for any pair of nodes* $e_1, e_2 \in E(\Gamma)$, *there exist a defining equation* F *of* M *such that* $E^{\mathrm{hor}}(F) = \{e_1, e_2\}$.

We note that in the latter case it follows that e_1 and e_2 are M–cross-related and thus by Theorem 1.1 the periods over the two corresponding vanishing cycles are proportional on M. What the theorem shows is that for divisorial degenerations there moreover exists a defining equation that only crosses this pair of nodes (see Theorem 1.10 also for the related results for affine invariant manifolds in the minimal stratum).

Enumerating strata as above that could contain irreducible components of ∂M is easy, and one can envision applying this to rule out the existence of certain linear subvarieties via degeneration analysis. Our most precise technical result in studying the equations and stratification of linear subvarieties is Proposition 3.11, which gives the coefficients of defining equations, starting from the basis for defining equations of M at p taken in reduced row echelon form with respect to a suitable homology basis.

Our proof will in fact yield a more general statement than the theorem above: for any linear subvariety of any boundary stratum D_Γ, the general points of its irreducible boundary components are contained in the strata $D_{\Gamma'}$ where Γ' is a purely horizontal or purely divisorial degeneration of Γ, obtained from Γ either by introducing one new vertical level, or by introducing a new collection of cross-related horizontal edges. This will allow us to recursively apply this theorem and thus navigate the boundary stratification of a linear subvariety.

The analytic structure near the boundary of linear subvarieties

Our next set of results provide some more detailed information about the geometry of a linear subvariety near its boundary. We recall that M is an immersed subvariety of the stratum, not an embedded one, and at its singular points we can only say that its local irreducible components are given by linear equations (this is simply to say that locally M looks like a finite union of linear subspaces). Similarly, when working with the closure $\overline{M}$ near its boundary point $p_0 \in \partial M$, we will work separately with the local irreducible components $\overline{Z}$ of $\overline{M}$ at p_0 and let $Z = \overline{Z} \cap M$.

Period coordinates on a stratum do not extend to the boundary; instead we have analytic *plumbing coordinates* in a neighborhood of the boundary. Using the precise information on the coefficients of defining equations for linear subvarieties obtained in Proposition 3.11, we can explicitly convert the linear equations in period coordinates into holomorphic equations in plumbing coordinates.

Theorem 1.6 *Let M be a linear subvariety and let $p_0 \in \partial M$. The local analytic equations for a local irreducible component $\overline{Z}$ of $\overline{M}$ near p_0 can be computed explicitly from the defining equations of M at a smooth point of Z. Analytically locally, $\overline{Z}$ is isomorphic to the product of $\mathbb{C}^n$ and varieties defined by binomial equations. In particular, Z is locally isomorphic to a (not necessarily normal) toric variety* (see (4-4) and (4-5)).

The strength of this result is that it allows us to describe the local structure of $\overline{Z}$ and of ∂Z near p_0 very precisely. Recall that an open stratum $D_\Gamma \cap U$ is contained in the closure of the open stratum $D_{\Gamma'} \cap U$ if and only if Γ' is an undegeneration of Γ. Any undegeneration is a composition of a *horizontal undegeneration*, which contracts some collection of horizontal edges of Γ, and a *vertical undegeneration*, which contracts a number of level transitions in Γ. The local defining equations allow us to show that certain such undegenerations occur in M.

Theorem 1.7 *If ∂M_Γ is nonempty, i.e. if $p_0 \in \partial M \cap D_\Gamma$, then, for any Γ' obtained from Γ by a composition of a vertical undegeneration and of a horizontal undegeneration that smooths some collection of M–cross-equivalence classes, the intersection $\partial M_{\Gamma'} = \partial M \cap D'_\Gamma$ is also nonempty.*

In [9] it is shown that the union D^{ver} of all open boundary strata D_Γ of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ such that Γ does not have horizontal nodes has simple normal crossings in $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. The above theorem says that in particular $\overline{M}$ is generically transverse to D^{ver}; a more precise statement is the following corollary of Theorem 1.6:

Corollary 1.8 *For any local irreducible component $\overline{Z}$ of $\overline{M}$ at p_0, if none of the defining equations of Z cross any horizontal nodes, then $\overline{Z}$ is smooth and $\partial Z \subset \overline{Z}$ is normal crossing.*

All periods of exact differentials over vanishing cycles are equal to 0, and thus horizontal nodes cannot arise when these are degenerated. This provides a very interesting situation, where all defining equations are nonhorizontal. The case of the double ramification locus will be treated by the first author in [5]. Here we consider the Hurwitz spaces of covers, the spaces of branched covers $f\colon X \to \mathbb{P}^1$ with prescribed branching over a number of points (see (4-7) for a precise definition). In Proposition 4.6 we use exact differentials to realize Hurwitz spaces as linear subvarieties of the strata and to construct a *smooth* compactification of Hurwitz spaces.

Cylinder deformation theorem

Given a boundary point $p_0 \in \partial M_\Gamma$ and a node $e \in E(\Gamma)$, it is impossible to choose a relative homology cycle crossing λ_e that varies continuously in U, as there is nontrivial monodromy. From the point of view of flat geometry, however, for any flat surface near p_0 one can naturally choose a "long cylinder" around the vanishing cycle λ_e. Wright has proven the fundamental cylinder deformation theorem, describing geometrically the types of deformations that can appear in affine invariant manifolds. We give a fundamentally new proof in a somewhat more general context.

Recall that two cylinders on a flat surface are called *parallel* if the periods of their circumference curves are real multiples of each other. For an affine invariant manifold $M \subseteq \Omega\mathcal{M}_{g,n}(\mu)$, cylinders $C_1, C_2 \subset X \in M$ are called *M–parallel* if they are parallel on X and parallel for all flat surfaces in a neighborhood of X in M. An equivalence class of M–parallel cylinders on X is a maximal collection $\mathcal{C} = \{C_1, \dots, C_d\}$ of cylinders on X that are pairwise M–parallel. The original intuition for the cylinder deformation theorem arose from the idea that affine invariant manifolds should be algebraic subvarieties (unproven at the time), which restricts the type of linear equations that are possible. However, the original proof used quite different methods, relying on deep results of Minsky and Weiss [15] and Smillie and Weiss [18] on the dynamics of the horocycle flow. Our proof follows the strategy of the original intuition.

Theorem 1.9 (cylinder deformation theorem) *Let M be an algebraic subvariety of a* ***meromorphic*** *stratum cut out by linear equations in period coordinates with* ***real*** *coefficients. Let $\mathcal{C} = \{C_1, \dots, C_d\}$ be an equivalence class of M–parallel horizontal cylinders on some $(X, \omega) \in M$. Then, for any $t, s \in \mathbb{R}$, the flat surface $a_t^{\mathcal{C}} u_s^{\mathcal{C}}(X, \omega)$ obtained by applying to each C_i the matrix*

$$a_t \circ u_s \quad \text{with } a_t = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \text{ and } u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},$$

and leaving the rest of the flat surface unchanged is also contained in M.

The above is a generalization of [21, Theorem 5.1], where it is assumed that M is an affine invariant manifold (living in a *holomorphic* stratum). We stress that to prove our results, we use degeneration techniques, only working near the boundary ∂M in $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$; note, though, that the theorem above applies at *any* point of M, not necessarily close to its boundary. Essentially what happens is that we know that M is cut out by linear equations near each of its points and, by analyzing the behavior of these equations near ∂M, we obtain sufficiently many necessary conditions on these equations in order to control deformations at each point of M.

The linear equations of affine invariant manifolds

Affine invariant manifolds are linear subvarieties of *holomorphic* strata, with all equations having *real* coefficients. Equivalently, by the foundational results of Eskin, Mirzakhani and Mohammadi, combined with the result of Filip, these are the (topological) closures of orbits of the $\mathrm{GL}^+(2,\mathbb{R})$–action on the holomorphic strata. These $\mathrm{GL}^+(2,\mathbb{R})$–orbits come up naturally in the study of billiards on rational polygons and the Teichmüller geodesic flow.

In this more restricted context of most interest we are able to obtain further information, similar to some results of Mirzakhani and Wright [16]. Of fundamental importance for us is the result of Avila, Eskin and Möller [1] that, for affine invariant manifolds, the tangent space projected to absolute homology is symplectic. This gives a way to use our precise understanding of relations among periods over vanishing cycles, given by Theorem 1.1, to obtain further results on top-horizontal-crossing equations. Our strongest result in this direction is for affine invariant manifolds in the minimal stratum:

Theorem 1.10 *Consider an affine invariant manifold M in the minimal stratum $\Omega\mathcal{M}_{g,1}(2g-2)$. Then*:

(1) *The space of defining equations of M is spanned by defining equations that cross at most two horizontal nodes.*

(2) *The space of defining equations of M that are linear combinations of periods over horizontal vanishing cycles is spanned by defining equations that are pairwise proportionalities of vanishing cycles.*

The two statements will follow from Propositions 6.4 and 6.5, respectively. Note that (2) above is the same statement as that of Theorem 1.5 for the case of horizontal divisorial degenerations — but in the context of affine invariant manifolds of the minimal stratum we prove it for arbitrary degenerations.

This precise description does not directly generalize to the case of the general stratum $\Omega\mathcal{M}_{g,n}(\mu)$, as we demonstrate in Examples 6.8 and 6.7. The main difficulty in discovering a suitable general statement lies in the fact that, while vanishing cycles are naturally elements of absolute homology group $H_1(X;\mathbb{Z})$, the defining equations naturally lie in $H_1(X,\underline{z};\mathbb{C})$ (recall that we are in the holomorphic case, so $\underline{p}=\varnothing$). We will investigate this general situation and application to classification of affine invariant manifolds in further work.

We also record some special properties of the boundary stratification of affine invariant manifolds in Proposition 6.9 and Corollary 6.10.

Outline of the paper

- In Section 2 we recall the moduli space of multiscale differentials, describe the setup and notation for our study of linear subvarieties via degenerations, and recall the relevant machinery and results of the first author from [4].
- In Section 3 we start by studying irreducible components of the boundary ∂M, proving Theorem 1.5, and then use this recursively to prove Theorem 1.1. We

then further study top-horizontal-crossing and nonhorizontal equations in detail, proving Theorems 1.2 and 1.4. Our most precise result is Proposition 3.11, which gives the coefficients of defining equations.

- In Section 4, we use these detailed results to focus on the nonemptiness and dimensions of the strata ∂M_Γ, converting linear equations to equations in plumbing coordinates to obtain Theorem 1.6. The form of the equations in plumbing coordinates yields Theorem 1.7, and allows us to construct a smooth compactification of Hurwitz spaces in Proposition 4.6.
- In Section 5 we further analyze the linear equations and degenerations to prove Theorem 1.9, our generalization of the cylinder deformation theorem.
- Finally, in Section 6 we specialize to the case of affine invariant manifolds. By [1], the tangent space of an affine invariant manifold, projected to absolute homology, is symplectic. We use this to prove Propositions 6.4 and 6.5, which together constitute Theorem 1.10.

Acknowledgments

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2 Notation and setup

In this section we recall those aspects of the setup, construction and results of [3] that we need, and also the setup and results of [4]. The most technical aspect of [4], log period spaces, is not necessary for our study.

Level graphs and multiscale differentials For a stable Riemann surface $(X, \underline{x}) \in \mathcal{M}_{g,n}$ we denote by Γ its dual graph. We will follow the convention of [3] in always suppressing the notation for marked points, unless they are used explicitly. A *level graph* structure $\overline{\Gamma}$ on Γ is given by a function $\ell\colon V(\Gamma) \twoheadrightarrow \{0, -1, \dots, -L(\Gamma)\}$. For any $i \in \{0, -1, \dots, -L(\Gamma)\}$, the subgraphs $\overline{\Gamma}_{(i)}$, $\Gamma_{(<i)}$ and so on are defined by taking the induced subgraph on the set of all vertices $v \in V(\Gamma)$ such that $\ell(v) = i$, or $\ell(v) < i$, and so on. For example, the vertices of $\overline{\Gamma}_{(i)}$ are all vertices at level i, and the edges are those edges $e \in E(\Gamma)$ both of whose endpoints lie at level i.

An edge $e \in E(\overline{\Gamma})$ is called *horizontal* if it connects two vertices of the same level, and called *vertical* otherwise, and we write $E(\overline{\Gamma}) = E^{\mathrm{hor}}(\overline{\Gamma}) \sqcup E^{\mathrm{ver}}(\overline{\Gamma})$. For a vertical edge $e \in E^{\mathrm{ver}}(\overline{\Gamma})$, we denote by $\ell(e^-)$ and $\ell(e^+)$ the levels of its bottom and top vertex, respectively. We denote by $E^{\mathrm{hor}}_{(i)}(\Gamma) \subseteq E^{\mathrm{hor}}(\Gamma)$ the set of horizontal edges connecting vertices of level i. A multiscale differential is the data of a stable Riemann surface X together with a collection $\eta = \{\eta_v\}$ of meromorphic differentials on the irreducible components X_v of X satisfying various conditions described in [2; 3] — in particular, η has simple poles at all horizontal nodes. An enhancement $\overline{\Gamma}^+$ of a level graph is a choice of a positive integer κ_e for every vertical edge e. This integer prescribes the order of zero of the multiscale differential to be $\kappa_e - 1$ at the top preimage of the node e, and the order of the pole to be $\kappa_e + 1$ at the bottom preimage of e. We will always work with level graphs with a chosen and fixed enhancement, but, to keep the notation manageable, from now on we will simply write Γ for an enhanced level graph. Additionally, the data of a multiscale differential includes a prong-matching, and great care is needed in understanding equivalence of multiscale differentials, but, as we will be working locally on $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, we will be able to mostly avoid these considerations.

Undegenerations and plumbing The boundary $\partial\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is stratified. It is convenient for us to denote by D_Γ the *open* boundary strata (note that in [9] this notation is used for closed boundary strata) indexed by enhanced level graphs. A stratum D_Γ is essentially a finite union of some finite covers of products of linear subspaces of products of some strata of meromorphic differentials; in particular, D_Γ may be disconnected (see [9, Section 4] and Remark 3.2 below for more discussion). All of our constructions will be performed locally in a neighborhood U, which we will now describe, of a chosen fixed point $p_0 = (X_0, \Gamma, \eta_0) \in D_\Gamma$.

The codimension of a stratum $\operatorname{codim}_{\Xi\overline{\mathcal{M}}_{g,n}(\mu)} D_\Gamma$ is equal to $H(\Gamma) + L(\Gamma)$, where $H(\Gamma) := \#E^{\mathrm{hor}}(\Gamma)$. Fix a small open neighborhood $p_0 \in W \subset D_\Gamma$. Then a neighborhood of p_0 in $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ can be given as $U := W \times \Delta^{H(\Gamma)+L(\Gamma)}$, where Δ is a sufficiently small complex disk around zero. Coordinates on the second factor are called plumbing coordinates, which we denote by $\{h_e\}_{e\in E^{\mathrm{hor}}(\Gamma)}$ and $\{t_i\}_{i\in\{-1,\dots,-L(\Gamma)\}}$. We will denote by $U^\circ := W \times (\Delta^*)^{H(\Gamma)+L(\Gamma)}$ the set of all smooth flat surfaces in U. From now on, when we speak of U and W, we will allow ourselves to further shrink the neighborhoods as necessary.

An open stratum $D_{\Gamma'}$ intersects U if and only if the (enhanced) level graph Γ' is an undegeneration of Γ (which we write as $\Gamma' \rightsquigarrow \Gamma$). Equivalently, there is a simplicial graph morphism $\mathrm{dg}\colon \Gamma \to \Gamma'$, which is obtained as a composition $\mathrm{dg} = \mathrm{dg}^{\mathrm{hor}} \circ \mathrm{dg}^{\mathrm{ver}}$ of the horizontal undegeneration $\mathrm{dg}^{\mathrm{hor}}$ that only contracts some set of horizontal edges, and a vertical undegeneration that only contracts some set of level transitions. We refer to [3] for a discussion of the behavior of enhancements and the (very delicate) behavior of prong-matchings under undegeneration. Explicitly, the closure $\overline{D}_{\Gamma'} \cap U$ is the coordinate subspace of U given by equations $h_e = 0$ for all $e \in E^{\mathrm{hor}}(\Gamma') \subseteq E^{\mathrm{hor}}(\Gamma)$ and $t_i = 0$ for all level transitions of Γ that persist in Γ'. From now on, whenever we speak of an undegeneration Γ', we implicitly mean with a *given* graph morphism $\mathrm{dg}\colon \Gamma \twoheadrightarrow \Gamma'$.

Any flat surface $p = (X, \omega) \in U^\circ$ can be obtained by plumbing some $(X_b, \eta_b) \in W \subset D_\Gamma$. The plumbing procedure replaces a neighborhood of each node $e \in X_b$, which is locally a union of two disks identified at the origin, with a cylinder, suitably glued to the rest of the surface. We denote by $\Lambda_e \subset X$ the pinching curve, also called the seam, which is the circumference curve of this cylinder. The *vanishing cycle* is the homology class $\lambda_e := [\Lambda_e] \in H_1(X \setminus \underline{z}, \underline{p}; \mathbb{Z})$. We recall that $H_1(X \setminus \underline{z}; \mathbb{Z}) \hookrightarrow H_1(X \setminus \underline{z}, \underline{p}; \mathbb{Z})$ (thinking of the vanishing cycles in relative, rather than absolute, homology will be essential in Section 6). The intersection pairing $H_1(X \setminus \underline{z}, \underline{p}; \mathbb{Z}) \times H_1(X \setminus \underline{p}, \underline{z}; \mathbb{Z}) \to \mathbb{Z}$ then allows us to compute intersection numbers of λ_e with elements of $H_1(X \setminus \underline{p}, \underline{z}; \mathbb{C})$. We note that λ_e is only defined up to sign; most of our formulas will include λ_e with coefficient proportional to the intersection number $\langle \gamma, \lambda_e \rangle$ for some $\gamma \in H_1(X \setminus \underline{p}, \underline{z}; \mathbb{C})$, which will eliminate this sign ambiguity. Cutting X along the multicurve $\Lambda := \{\Lambda_e\}_{e \in E(\Gamma)}$ decomposes the smooth Riemann surface X into the union $X_{(0)} \cup X_{(-1)} \cup \cdots \cup X_{(-L(\Gamma))}$ of its levelwise pieces, where the pieces intersect along the seams Λ_e for $e \in E^{\mathrm{ver}}(\Gamma)$.

While, as discussed above, the local coordinates on $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ transverse to D_Γ are given by $t_{-1}, \dots, t_{-L(\Gamma)}$ and $\{h_e\}_{e \in E^{\mathrm{hor}}(\Gamma)}$, the plumbing coordinates s_e for vertical nodes are related to t_i by the equation

$$s_e = \prod_{i=\ell(e^-)}^{\ell(e^+)-1} t_i^{m_{e,i}}, \tag{2-1}$$

where we recall that, by definition [3, (6.7)], a_i is the least common multiple of κ_e for all $e \in E^{\mathrm{ver}}(\Gamma)$ such that $\ell(e^+) > i \geq \ell(e^-)$, and $m_{e,i} := a_i / \kappa_e$.

The boundary neighborhood in M We fix once and for all a linear subvariety $M \subseteq \Omega\mathcal{M}_{g,n}(\mu)$ of codimension m, and will consider its closure $\overline{M} \subseteq \Xi\overline{\mathcal{M}}_{g,n}(\mu)$,

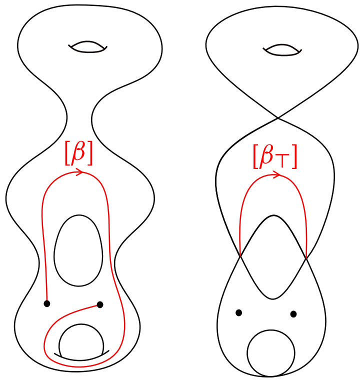


Figure 2: The top-level restriction of a homology class.

so that the projectivization $\mathbb{P}\overline{M} \subseteq \mathbb{P}\,\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is compact. Since $\overline{M}$ is the closure of an algebraic subvariety $M \subseteq \Omega\mathcal{M}_{g,n}(\mu)$ in the algebraic compactification $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ of $\Omega\mathcal{M}_{g,n}(\mu)$, it follows that $\overline{M}$ is algebraic [17, Corollary 10.1].

We will choose $p_0 \in \partial M \cap D_\Gamma$ and, for any undegeneration $\Gamma' \rightsquigarrow \Gamma$, let $\partial M_{\Gamma'} := \partial M \cap U \cap D_{\Gamma'} \subset D_{\Gamma'}$. Recall that in general M is an immersed submanifold of $\Omega\mathcal{M}_{g,n}$, and we will always want to work at a flat surface $p = (X, \omega)$ that is a smooth point of $M \cap U$, to avoid having to deal with M having multiple local irreducible components at p (each linear in period coordinates). We use (X', Γ', η') to denote points $(X', \omega') \in \partial M_{\Gamma'}$ on the (local) open strata corresponding to undegenerations $\Gamma' \rightsquigarrow \Gamma$. We will often omit ω or ω' in our notation for flat surfaces.

Top level of paths and homology classes In [4], the first author determined the defining equations for $\partial M_\Gamma \subseteq D_\Gamma$ at p_0, in (generalized) period coordinates on D_Γ, starting from the defining equations for $M \subseteq \Omega\mathcal{M}_{g,n}(\mu)$ at a nearby point $p \in M \cap U$. Qualitatively, the result is that ∂M_Γ is given by linear equations on D_Γ, but we will need the precise description of these equations, which we now recall.

To state the results of [4], we need to restrict paths in X to their top level. The top level $\top(\beta)$ of any collection of paths $\beta \subset X$ is the largest i such that $\beta \cap X_{(i)} \neq \varnothing$. The *top level* $\top([\beta])$ of a class $[\beta] \in H_1(X \setminus \underline{p}, \underline{z}; \mathbb{C})$ is the minimum of $\top(\beta)$ over all collections of paths β representing the class $[\beta]$. For a homology class $[\beta]$, we define its *top-level restriction* $[\beta_\top]$ to be the element $[\beta_\top] \in H_1(X_{(\top(\beta))} \setminus (\underline{p} \cup \Lambda^{\mathrm{hor}}_{(\top(\beta))}), \underline{z} \cup \Lambda^{\mathrm{ver},+}_{(\top(\beta))}; \mathbb{Z})$ defined by choosing a collection of paths β representing $[\beta]$ such that $\top(\beta) = \top([\beta])$, and restricting each path in β to $X_{(\top([\beta]))}$, considered as a relative homology class there. In [4, Proposition 4.2], it is shown that this is well defined. See Figure 2.

For a smooth flat surface $p \in U^\circ$ a homology class $[\beta] \in H_1(X \setminus \underline{p}, \underline{z}; \mathbb{C})$ is said to be *crossing* a node $e \in E(\Gamma)$ if $\langle [\beta], \lambda_e \rangle \neq 0$, where recall that we think of λ_e as an element of $H_1(X \setminus \underline{z}, \underline{p}; \mathbb{Z})$. We call $[\beta]$ a *top-horizontal-crossing cycle* if it crosses some horizontal vanishing cycle *at level* $\top([\beta])$. To simplify language, we will call *non-top-horizontal-crossing* any class $[\beta]$ that is not a top-horizontal-crossing cycle, and emphasize that such a non-top-horizontal-crossing $[\beta]$ may still intersect horizontal vanishing cycles at levels strictly below $\top([\beta])$.

A Γ–adapted basis Recall from [4] that a Γ–adapted basis is a basis for $H_1(X \setminus \underline{p}, \underline{z}; \mathbb{Z})$ satisfying the following properties. First, all of its elements that are top-horizontal-crossing cycles have intersection 1 with λ_e for a unique $e \in E^{\mathrm{hor}}(\Gamma)$, where these e are distinct for different top-horizontal-crossing cycles in the basis and do not cross any other horizontal nodes. Elements of a Γ–adapted basis that have top level i can be listed as

$$\{\delta_1^{(i)}, \dots, \delta_{c(i)}^{(i)}, \alpha_1^{(i)}, \dots, \alpha_{d(i)}^{(i)}\},$$

where each $\delta_j^{(i)}$ is a top-horizontal-crossing cycle with $\langle \delta_j^{(i)}, \lambda_{e_j^{(i)}} \rangle = 1$ for some distinct horizontal node $e_j^{(i)} \in E_{(i)}^{\mathrm{hor}}(\Gamma)$, and such that $\delta_j^{(i)}$ does not cross any other horizontal nodes at *any* level. Furthermore, the definition of being a Γ–adapted basis requires that each $\alpha_j^{(i)}$ does not cross any horizontal nodes at any level and that, for any i, the top-level restrictions $\{[(\alpha_1^{(i)})_\top], \dots, [(\alpha_{d(i)}^{(i)})_\top]\}$ form a basis of the quotient of $H_1(X_{(i)} \setminus (\underline{p} \cup \Lambda_{(i)}^{\mathrm{hor}}), \underline{z} \cup \Lambda_{(i)}^{\mathrm{ver},+}; \mathbb{Z})$ by the subspace of global residue conditions. The existence of a Γ–adapted basis for any Γ is proven in [4, Proposition 4.8]. Sometimes we do not need to specify the level of the homology classes or whether they cross horizontal nodes or not. In this case we write the Γ–adapted basis simply as

$$\{\gamma_1, \dots, \gamma_K\} = \bigsqcup_{i=-L(\Gamma)}^{0} \{\delta_1^{(i)}, \dots, \delta_{c(i)}^{(i)}, \alpha_1^{(i)}, \dots, \alpha_{d(i)}^{(i)}\}, \tag{2-2}$$

where $K := \dim H_1(X \setminus \underline{p}, \underline{z}; \mathbb{C}) = \dim \Xi\overline{\mathcal{M}}_{g,n}(\mu)$. We will choose and fix a Γ–adapted basis from now on.

Defining equations of M The technical core of our arguments is investigating the linear equations for ∂M_Γ. To keep the notation manageable, we simply say that $F \in H_1(X \setminus \underline{p}, \underline{z}; \mathbb{C})$ is a *defining equation of M* if $\int_F \omega = 0$ holds identically on M in a neighborhood of a fixed chosen flat surface $p \in M \cap U$. We will denote by $N \subseteq H_1(X \setminus \underline{p}, \underline{z}; \mathbb{C})$ the linear space of all defining equations of M at p, denoted thus because it is the normal space in period coordinates. As discussed in the introduction, the space N is locally constant along M near p, and thus throughout the paper we should be carefully treating irreducible components Z of $M \cap U$ (which, after shrinking U, are in bijection with the local irreducible components of $\overline{M}$ at p_0) individually. To keep the notation and language manageable, we will just speak of defining equations, making the discussion of local irreducible components precise in Section 4, where it is crucial.

Denote by $C_l \in \mathbb{C}$ the coefficients of F in our fixed Γ–adapted basis $\{\gamma_l\}_{l=1,\dots,K}$, so that

$$F(X,\omega) = \sum_{l=1}^{K} C_l \int_{\gamma_l} \omega. \tag{2-3}$$

Equivalently, writing out the basis elements separately, we denote the coefficients of F by $A_l^{(i)}, B_l^{(i)} \in \mathbb{C}$, so that

$$F(X,\omega) = \sum_{i=-L(\Gamma)}^{\top(F)} \left(\sum_{l=1}^{c(i)} A_l^{(i)} \int_{\delta_l^{(i)}} \omega + \sum_{l=1}^{d(i)} B_l^{(i)} \int_{\alpha_l^{(i)}} \omega \right). \tag{2-4}$$

Writing down all defining equations of M at p involves a choice of the basis of the vector space N. We will always choose a basis of defining equations such that the matrix $C = (C_{kl})$ of the coefficients of defining equations (2-3) is in *reduced row echelon form* (*rref*) with respect to our chosen Γ–adapted basis, and denote by $F_1, \dots, F_m$ such an *rref basis*.

Equations of ∂M_Γ from equations of M In [4], the main quantitative result is a way to read off the equations for $\partial M_\Gamma \subseteq D_\Gamma$ from an rref basis:

Theorem 2.1 [4, Theorem 1.2 and Proposition 8.2] *For each $j = 1, \dots, m$, if F_j is a nonhorizontal cycle, let $G_j := [(F_j)_\top]$, and if F_j is a top-horizontal-crossing cycle, then let $G_j := 0$. Then $G_1, \dots, G_m$ form a basis for the space of local defining equations for ∂M_Γ within D_Γ.*

Essentially what this says is that we represent each equation F_j by a collection of paths whose top level is minimal possible, equal to $\top(F_j)$; then, if this collection of paths crosses any horizontal node at its top level, then on D_Γ we "lose" this defining equation F_j; otherwise, the equation F_j on D_Γ yields the equation $[(F_j)_\top]$.

In view of this theorem, for our fixed linear subvariety M and for any undegeneration $\Gamma' \rightsquigarrow \Gamma$ we denote by $c(\Gamma')$ the number of defining equations of M at p that are lost on $D_{\Gamma'} \cap U$. Thus, the number of defining equations for $\partial M_{\Gamma'}$ inside $D_{\Gamma'}$ is equal to $m - c(\Gamma')$, and thus Theorem 2.1 implies that

$$\operatorname{codim}_{\Xi\overline{\mathcal{M}}_{g,n}(\mu)}(\partial M_{\Gamma'}) = H(\Gamma') + L(\Gamma') + m - c(\Gamma'), \tag{2-5}$$

since $\partial M_{\Gamma'}$ has codimension $m - c(\Gamma')$ within the open stratum $D_{\Gamma'}$, which itself has codimension $H(\Gamma') + L(\Gamma')$ in $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. For further use, we call an undegeneration $\mathrm{dg}\colon \Gamma \twoheadrightarrow \Gamma'$ *divisorial* if $\dim_{\mathbb{C}} \partial M_\Gamma = \dim_{\mathbb{C}} \partial M_{\Gamma'} - 1$.

Remark 2.2 A consequence of Theorem 2.1 is that linear equations for ∂M_Γ can be *lifted* to M. More precisely, if F is a linear equation among periods which is satisfied on ∂M_Γ in a neighborhood of p_0, and F is completely contained in level i, i.e. F can be represented by paths contained in $X_{(i)}$, then there exists a linear equation G for M, valid in a neighborhood of a nearby point $(X, \omega) \in M$, such that $G_\top = F$. In other words, F is the top-level restriction of the linear equation G. We stress that one can only lift an equation F for ∂M_Γ if it is completely contained in a fixed level. Any linear equation defining ∂M_Γ can then be written as a sum of linear equations, each of which is completely contained in some level (these levels might be different for different summands), and each of these summands can be lifted. ◁

3 Degenerations of linear equations

In what follows, given a defining equation F of M at p, it will be useful to consider various associated periods. For any undegeneration $\Gamma' \rightsquigarrow \Gamma$ and for any collection of integers $\{n_e : e \in E(\Gamma')\}$, we define the $(\Gamma', \underline{n})$*–residue of* F by

$$R(F, \Gamma', \underline{n}) := \sum_{e \in E(\Gamma')} n_e \langle F, \lambda_e \rangle \int_{\lambda_e} \omega. \tag{3-1}$$

Proposition 3.1 (the monodromy argument [4, Proposition 7.6]) *For any defining equation F of M at p, if $\partial M_{\Gamma'}$ is nonempty for some undegeneration $\Gamma' \rightsquigarrow \Gamma$, then, for some collection $\underline{n}$ of **positive** integers, the residue $R(F, \Gamma', \underline{n})$ is identically zero on M.*

For the convenience of the reader, we quickly recall from [4] the outline of the proof.

Proof Let $f : \Delta \to \overline{M}$ be a holomorphic map from a disk such that $f(\Delta^*)$ is contained in the smooth locus of M, $p \in f(\Delta)$ and $f(0) = p_0$. We define n_e to be the (positive since $s_e(p_0) = 0$) vanishing order of $s_e \circ f$ at $z = 0 \in \Delta$, where we recall that s_e

is the plumbing parameter for the corresponding node. Using Picard–Lefschetz, the monodromy of any cycle $[\beta] \in H_1(X \setminus \underline{p}, \underline{z}; \mathbb{Z})$ for the Gauss–Manin connection along f can be computed in terms of vanishing cycles as

$$[\beta] \mapsto [\beta] + \sum_{e \in E(\Gamma)} n_e \langle \gamma, \lambda_e \rangle \lambda_e.$$

Thus, parallel transport along the generator of $\pi_1(\Delta^*)$ transforms the equation F written in the form (2-3) into

$$\sum_{l=1}^{K} C_l \left(\int_{\gamma_l} \omega + \sum_{e \in E(\Gamma)} n_e \langle \gamma_l, \lambda_e \rangle \int_{\lambda_e} \omega \right),$$

which must then also be a defining equation of M at p, and then subtracting F in the form (2-3) from this equation yields the proposition. □

We now use this monodromy argument to quickly prove the necessary conditions for boundary strata of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ to contain irreducible components of the boundary ∂M. Since $\overline{M} \subseteq \Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is an algebraic subvariety, its intersection ∂M with the boundary of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, which is a divisor, is an equidimensional variety of dimension $\dim_{\mathbb{C}} \partial M = \dim_{\mathbb{C}} M - 1$. For $U \ni p_0$ sufficiently small, each irreducible component Y of $\partial M \cap U$ must contain p_0. Thus, the generic point of Y must be contained in the open stratum $D_{\Gamma'}$ for some undegeneration $\Gamma' \rightsquigarrow \Gamma$.

Proof of Theorem 1.5 Denote by $Y^o := Y \cap D_{\Gamma'}$ the open part of a divisorial boundary component, so that $\dim_{\mathbb{C}} Y^o = \dim_{\mathbb{C}} M - 1$. Substituting the dimension of Y^o from (2-5) yields

$$H(\Gamma') + L(\Gamma') + m - c(\Gamma') = m + 1. \tag{3-2}$$

Recall that $c(\Gamma')$ is the number of equations F_j in the rref basis that are "lost" on $\partial M_{\Gamma'}$, which are those where F_j is top-horizontal-crossing. Proposition 3.1 shows that every equation F_j crosses at least two horizontal nodes (otherwise, if it only crossed one horizontal node, then the period of ω over the corresponding vanishing cycle would vanish, which is impossible, since the twisted differential must have a simple pole at every horizontal node). By definition, for each top-horizontal-crossing equation F_j of the rref basis, the pivot corresponds to a horizontal node λ_e crossed by F_j. Thus, either $H(\Gamma') = 0$ or $c(\Gamma') \le H(\Gamma') - 1$. In the first case we conclude from (3-2) that $L(\Gamma') = 1$.

In the latter case, substituting $H(\Gamma') \ge c(\Gamma') + 1$ into the left-hand-side of (3-2) yields $c(\Gamma') + 1 + L(\Gamma') + m - c(\Gamma') \ge m + 1$, which is only possible if $L(\Gamma') = 0$ and, moreover, if $H(\Gamma') = c(\Gamma') + 1$. In that case the set of equations of the rref basis of defining equations can cross only one additional horizontal node in addition to the pivots, and the matrix, in rref, of defining equations of M must have the form

$$\left(\begin{array}{ccccc|c} 1 & 0 & \dots & 0 & d_1 & \\ 0 & \ddots & \vdots & \vdots & \vdots & \\ \vdots & 0 & 1 & 0 & d_{H(\Gamma')-1} & * \\ 0 & 0 & 0 & 1 & d_{H(\Gamma')} & \\ \hline & & \mathbf{0} & & & * \end{array}\right),$$

where the upper rows correspond to top-horizontal-crossing equations for Γ', the d_l are nonzero and the lower rows correspond to nonhorizontal equations for Γ'. □

3.1 Generalized strata, and constructing degenerations recursively

In the proof of Theorem 1.1 below, and for potential applications of our machinery to classifying or ruling out existence of linear subvarieties of a given stratum, one needs to apply Theorem 1.5 recursively. Starting from a noncomplete linear subvariety $M \subseteq \Omega\mathcal{M}_{g,n}(\mu)$, we consider a divisorial boundary component $M' := \partial M_\Gamma \subseteq D_\Gamma$, for which Theorem 1.5 gives necessary conditions on the graph Γ. By the results of [4], M' is locally given within D_Γ by linear equations, and we would like to apply Theorem 1.5 again to yield a further divisorial degeneration of M', assuming again that M' is noncomplete. However, Theorem 1 of [4] as stated does not apply to show that $\partial M'_{\Gamma'}$ inside of $\overline{D}_{\Gamma'}$ is a linear subvariety, because in general $\overline{D}_{\Gamma'}$ will be singular.

Remark 3.2 (generalized strata of differentials) The stratification of the boundary of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is discussed in detail in [9, Section 4]. The boundary strata are called there *generalized* strata of differentials. We now recall their geometric description and explain how our results can be adapted to this generality.

Let $\boldsymbol{g} = (g_1, \dots, g_k)$ be a tuple of genera, $\boldsymbol{n} = (n_1, \dots, n_k)$ a tuple of positive integers and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$ a tuple of types of differentials, i.e. μ_i is a partition of $2g_i - 2$ into (not necessarily positive) integers of length n_k. The disconnected stratum is defined to be

$$\Omega\mathcal{M}_{\boldsymbol{g},\boldsymbol{n}}(\boldsymbol{\mu}) := \prod_{i=1}^{k} \Omega\mathcal{M}_{g_i,n_i}(\mu_i),$$

and the projectivization $\mathbb{P}\Omega\mathcal{M}_{\boldsymbol{g},\boldsymbol{n}}(\boldsymbol{\mu})$ is the quotient of $\Omega\mathcal{M}_{\boldsymbol{g},\boldsymbol{n}}(\boldsymbol{\mu})$ by the diagonal $\mathbb{C}^*$–action. A residue subspace $\mathfrak{R}$ is a set of linear equations on residues, modeled on the global residue conditions and matching residue conditions; see [9, Section 4.1] for the precise definition. The generalized stratum $\Omega\mathcal{M}^{\mathfrak{R}}_{\boldsymbol{g},\boldsymbol{n}}(\boldsymbol{\mu})$ modeled on a residue subspace $\mathfrak{R}$ is the subspace of $\Omega\mathcal{M}_{\boldsymbol{g},\boldsymbol{n}}(\boldsymbol{\mu})$ consisting of all surfaces with residues lying in $\mathfrak{R}$. In [9, Proposition 4.2] the authors construct a compactification $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathfrak{R}}_{\boldsymbol{g},\boldsymbol{n}}(\boldsymbol{\mu})$ of $\mathbb{P}\Omega\mathcal{M}^{\mathfrak{R}}_{\boldsymbol{g},\boldsymbol{n}}(\boldsymbol{\mu})$ similar to the moduli space of multiscale differentials. For an enhanced level graph Γ and for each level i, let $(\boldsymbol{g}^{[i]}, \boldsymbol{n}^{[i]}, \boldsymbol{\mu}^{[i]}, \mathfrak{R}^{[i]})$ be the tuple consisting of the genera, number of points, types of differentials and residues conditions at each irreducible component of level i. The *generalized stratum* associated to Γ is

$$B_\Gamma := \Omega\mathcal{M}^{\mathfrak{R}^{[0]}}_{\boldsymbol{g}^{[0]},\boldsymbol{n}^{[0]}}(\boldsymbol{\mu}^{[0]}) \times \prod_{i=-L(\Gamma)}^{-1} \mathbb{P}\Omega\mathcal{M}^{\mathfrak{R}^{[i]}}_{\boldsymbol{g}^{[i]},\boldsymbol{n}^{[i]}}(\boldsymbol{\mu}^{[i]})$$

and, by replacing $\mathbb{P}\Omega\mathcal{M}^{\mathfrak{R}^{[i]}}_{\boldsymbol{g}^{[i]},\boldsymbol{n}^{[i]}}(\boldsymbol{\mu}^{[i]})$ with $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathfrak{R}^{[i]}}_{\boldsymbol{g}^{[i]},\boldsymbol{n}^{[i]}}(\boldsymbol{\mu}^{[i]})$, we define $\overline{B}_\Gamma$ similarly. The generalized stratum B_Γ admits a system of *generalized period coordinates* as described in [4, Section 2.6], with transition functions that are linear on the top level and projective-linear on lower levels.

In [9], the authors construct the diagram

$$\begin{array}{ccccccc} & & & D^s_\Gamma & & & \\ & & \swarrow^{p_\Gamma} & & \searrow^{c_\Gamma} & & \\ \overline{B}_\Gamma & \hookleftarrow & B_\Gamma & & & D_\Gamma & \hookrightarrow & \overline{D}_\Gamma \end{array}$$

where c_Γ and p_Γ are covering maps. We do not give the precise definition of D^s_γ, c_Γ and p_Γ and instead refer the reader to [9, Section 4]. ◁

The main theorem of [4] can then be rephrased as:

Proposition 3.3 *Let $M \subseteq \Omega\mathcal{M}_{g,n}(\mu)$ be a linear subvariety. Then*

$$p_\Gamma(c_\Gamma^{-1}(\partial M_\Gamma)) \subseteq B_\Gamma$$

is a ***levelwise*** *linear subvariety for the linear structure on B_Γ.*

By abuse of notation, we will just write ∂M_Γ for $p_\Gamma(c_\Gamma^{-1}(\partial M_\Gamma))$. Levelwise here means that the linear equations defining ∂M_Γ only restrict periods of the same top level. Now, given a linear subvariety M' of B_Γ and a boundary stratum $D_{\Gamma'}$ of $\overline{B}_\Gamma$, we proceed in the same way and can thus consider $\partial M'_{\Gamma'} \subseteq D_{\Gamma'}$ also as a linear subvariety, by the same abuse of notation as above.

Remark 3.4 (constructing chains of divisorial degenerations) By the previous remark, we can now use Theorem 1.5 to construct chains of undegenerations, where each is divisorial in the next. Let M be a linear subvariety in a (possibly generalized) stratum B and $p_0 \in \partial M_\Gamma$ a boundary point. The undegenerations $\Gamma' \rightsquigarrow \Gamma$ corresponding to boundary divisors $D_{\Gamma'} \subseteq B$ are those where Γ' either has only two levels and no horizontal nodes or has a unique edge which is horizontal. In the former case, the undegeneration $\Gamma' \rightsquigarrow \Gamma$ corresponds to keeping only those edges of Γ that cross some level transition (i.e. such that $\ell(e^+) > i \geq \ell(e^-)$). Since $p_0 \in \partial M_\Gamma \subseteq \overline{\partial M}_{\Gamma'}$, in either case the intersection of $\overline{M}$ with $\overline{D}_{\Gamma'}$ is nonempty. Since it is an intersection with a divisor, it follows that $\dim \overline{\partial M}_{\Gamma'} = \dim M - 1$.

If $D_{\Gamma'}$ is a purely vertical divisorial stratum, by Theorem 1.5, a generic point of the intersection of $\overline{M}$ with $\overline{D}_{\Gamma'}$ is contained in the open boundary stratum $D_{\Gamma'}$.

On the other hand, if $D_{\Gamma'}$ is a purely horizontal boundary stratum, again by Theorem 1.5 there exists an intermediate undegeneration $\Gamma' \rightsquigarrow \Gamma'' \rightsquigarrow \Gamma$ such that some irreducible component of ∂M is generically contained in $\partial M_{\Gamma''}$, $\dim_{\mathbb{C}} \partial M_{\Gamma''} = \dim_{\mathbb{C}} M - 1$ and furthermore Γ'' is a purely horizontal level graph.

We can thus construct chains of undegenerations

$$\mathrm{pt} = \Gamma_0 \rightsquigarrow \Gamma_1 \rightsquigarrow \cdots \rightsquigarrow \Gamma_d = \Gamma$$

such that each boundary stratum ∂M_{Γ_i} is nonempty, each undegeneration

$$\Gamma_j \rightsquigarrow \Gamma_{j+1}$$

is either purely vertical or purely horizontal and furthermore

$$\dim_{\mathbb{C}} \partial M_{\Gamma_{j+1}} = \dim_{\mathbb{C}} \partial M_{\Gamma_j} - 1.$$

Note that pt, a single vertex and no edges, corresponds to the open stratum B itself. Such a chain of divisorial degenerations is determined by prescribing at each step j whether the undegeneration $\Gamma_j \rightsquigarrow \Gamma_{j+1}$ smooths some given level transition (and nothing else) or smooths a given horizontal edge. In the latter case the undegeneration may also have to smooth some further collection of horizontal edges.

Sometimes it will be more convenient to think of degenerations rather than undegenerations. When thinking of $\Gamma_j \rightsquigarrow \Gamma_{j+1}$ as a degeneration of Γ_j, instead of smoothing out a level transition or a collection of nodes we will then say that the degeneration pinches a level transition or a collection of nodes. $\lhd$

Proof of Theorem 1.2 Consider the closed boundary divisor $\bar{D}_{\Gamma[i]} := \{t_i = 0\} \subseteq \partial \Xi \overline{\mathcal{M}}_{g,n}(\mu)$. In other words, $\Gamma[i]$ is the undegeneration of Γ opening up all horizontal nodes and all level passages except the one between level $i+1$ and level i. In particular, $E(\Gamma[i]) := \{e \in E(\Gamma) : \ell(e^+) > i \geq \ell(e^-)\}$. Since the intersection $\bar{D}_{\Gamma[i]} \cap \overline{M}$ is nonempty because it contains p_0, this intersection is a divisor in $\overline{M}$. Let Y be an irreducible component of ∂M contained in $\overline{\partial M}_{\Gamma[i]}$. By Theorem 1.5, Y is generically contained in $D_{\Gamma[i]}$. Then let F be any defining equation of M, written in the form (2-3). Since $\partial M_{\Gamma[i]}$ is nonempty, it follows from Proposition 3.1 that there exist positive integers n_e for $e \in E(\Gamma[i])$ such that

$$\sum_{e \in E(\Gamma[i])} n_e \langle F, \lambda_e \rangle \int_{\lambda_e} \omega = 0. \tag{3-3}$$

We recall that the integers n_e are computed as vanishing orders of the plumbing parameter s_e along a degenerating family and thus by (2-1) there exists an integer d such that

$$n_e = d \cdot m_{e,i},$$

and thus (3-3) is equivalent to

$$R_i(F) := \sum_{e \in E(\Gamma[i])} m_{e,i} \langle F, \lambda_e \rangle \int_{\lambda_e} \omega = 0. \tag{3-4}$$

□

3.2 *M*–cross-related nodes, and proofs of Theorems 1.1 and 1.4

We now further investigate the form of linear equations crossing horizontal nodes, setting up the notation and preliminary results for the proof of Theorems 1.1 and 1.4.

Definition 3.5 For a defining equation F, we let

$$E^{\mathrm{hor}}(F) := \{e \in E^{\mathrm{hor}}(\Gamma) \mid \langle F, \lambda_e \rangle \neq 0\}$$

be the set of horizontal nodes crossed by F. A set $S \subseteq E^{\mathrm{hor}}(\Gamma)$ is called *M–correlated* if $S = E^{\mathrm{hor}}(F)$ for some defining equation F of M at p. An M–correlated set S is called *M–primitive* if no proper subset of S is M–correlated. We let $\sim$ be the equivalence relation on $E^{\mathrm{hor}}(\Gamma)$ generated by M–primitive subsets and, if $e \sim e'$, we say e and e' are *M–cross-related*. In words, two nodes e and $e' \in E^{\mathrm{hor}}(\Gamma)$ are M–cross-related if there exist M–primitive collections $S_1, \dots, S_k$ of horizontal nodes, and a sequence $e = e_0, \dots, e_k = e'$ of horizontal nodes such that $\{e_i, e_{i+1}\} \subseteq S_{i+1}$ for $i = 0, \dots, k-1$. The relation $\sim$ partitions $E^{\mathrm{hor}}(\Gamma)$ into *M–cross-equivalence classes*. ◁

Remark 3.6 The purpose of this definition is to formalize the notion that there is a defining equation F that crosses both nodes and that cannot be written as a sum of two defining equations that each cross a strictly smaller collection of horizontal nodes. Note that the definition does not require M–cross-related nodes to be of the same level, but, as periods of ω over horizontal vanishing cycles of different levels go to zero at different rates, we will see below in Corollary 3.9 that M–cross-related horizontal nodes must in fact have the same level. ◁

We now show that M–cross-equivalence classes can be computed using the rref basis $(F_1, \dots, F_m)$. We say two nodes $e, e' \in E^{\mathrm{hor}}(\Gamma)$ are *rref-cross-related* if there exists a chain of elements $F_{l_1}, \dots, F_{l_k}$ of the rref basis and a sequence $e = e_0, \dots, e_k = e'$ of horizontal nodes such that $\{e_i, e_{i+1}\} \subseteq E^{\mathrm{hor}}(F_{l_{i+1}})$ for $i = 0, \dots, k-1$. Said differently, rref-cross-equivalence is the equivalence relation generated by $E^{\mathrm{hor}}(F_1), \dots, E^{\mathrm{hor}}(F_m)$.

Lemma 3.7 *Two horizontal nodes are M–cross-related if and only if they are rref-cross-related.*

Proof We first claim that the set $E^{\mathrm{hor}}(F_j)$ is M–primitive for each $j = 1, \dots, m$. Assume for contradiction that there exists a defining equation F with $E^{\mathrm{hor}}(F) \subsetneq E^{\mathrm{hor}}(F_j)$. If we write $F = \sum_{l=1}^{m} a_l F_l$ in terms of the rref basis, then assume for contradiction that $a_l \neq 0$ for some $l \neq j$. But then $E^{\mathrm{hor}}(F)$ must contain the horizontal node corresponding to the pivot of F_l, which is not contained in $E^{\mathrm{hor}}(F_j)$. This gives a contradiction, and thus rref-cross-equivalence implies M–cross-equivalence.

For the other direction, assume that $E^{\mathrm{hor}}(F)$ is M–primitive and $e, e' \in E^{\mathrm{hor}}(F)$. Denote by E_0 the rref-equivalence class of e, and reorder the rref basis so that $E^{\mathrm{hor}}(F_j) \subseteq E_0$ for $j \leq u$ (for some u) and $E^{\mathrm{hor}}(F_j) \cap E_0 = \varnothing$ for $j > u$, and again write F as $F = \sum_{l=1}^{m} a_l F_l$.

We claim that $E^{\mathrm{hor}}\big(\sum_{l=1}^{u} a_l F_l\big) \subseteq E^{\mathrm{hor}}(F)$. Note that $E^{\mathrm{hor}}\big(\sum_{l=1}^{u} a_l F_l\big) \subseteq E_0$ by construction. If the claim were false, there would exist a node crossed by $\sum_{l=1}^{u} a_l F_l$ but not by F. But then one of the F_j with $j > u$ has to cross a node in E_0, which is impossible.

Since $E^{\mathrm{hor}}(F)$ is M–primitive we conclude that $E^{\mathrm{hor}}(F) = E^{\mathrm{hor}}\big(\sum_{l=1}^{u} a_l F_l\big)$ and, by construction, any pair of nodes in $E^{\mathrm{hor}}\big(\sum_{l=1}^{u} a_l F_l\big)$ is rref-related. □

We now have the tools to prove Theorem 1.1, which says that M–cross-related horizontal vanishing cycles have proportional periods. Before doing this, we illustrate what this result means in a simple example.

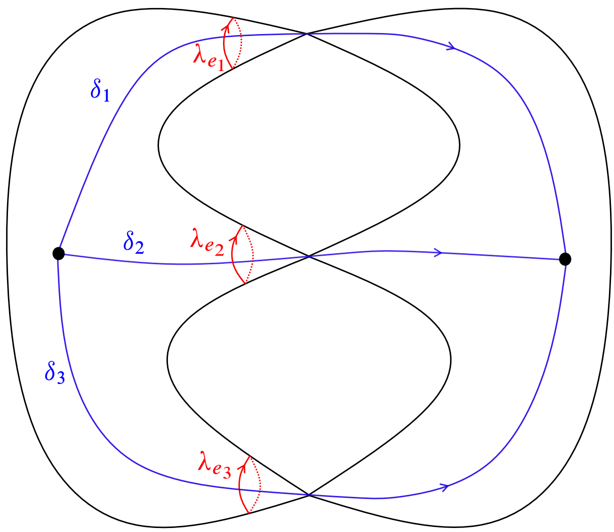


Figure 3: A point in the boundary of $\Omega\mathcal{M}_{2,2}(1,1)$, with three horizontal nodes.

Example 3.8 (M–cross-related nodes) Suppose that we have a linear submanifold M in the genus 2 stratum $\Omega\mathcal{M}_{2,2}(1,1)$, and that $p_0 \in \partial M$ is as shown in Figure 3, with three horizontal nodes e_1, e_2 and e_3 and no other nodes. We suppose that one of the defining equations of M at p is

$$c_1 \int_{\delta_1} \omega + c_2 \int_{\delta_2} \omega + c_3 \int_{\delta_3} \omega = 0,$$

where the c_i are nonzero complex numbers. And suppose that there is no other defining equation that crosses a proper subset of the horizontal vanishing cycles $\{\lambda_{e_1}, \lambda_{e_2}, \lambda_{e_3}\}$. Then the nodes e_1, e_2 and e_3 are M–cross-related, and Theorem 1.1 implies that the periods of ω over λ_{e_i} are all proportional on M.

There in fact exist affine invariant manifolds that locally have the above description near certain boundary points. For instance, one can take the eigenform locus E_D (discovered, independently, by Calta [6] and McMullen [14]) for $D \equiv 0, 1 \bmod 4$ a nonsquare positive integer. ◁

We are now ready to prove proportionality of periods over vanishing cycles for M–cross-related nodes.

Proof of Theorem 1.1 The idea of the proof is to use Theorem 1.5 recursively to construct a suitable chain of divisorial undegenerations

$$\mathrm{pt} = \Gamma_0 \rightsquigarrow \Gamma_1 \rightsquigarrow \cdots \rightsquigarrow \Gamma_d = \Gamma.$$

As explained in Remark 3.4, many such chains can be constructed, and to specify a chain we need to specify at each step either a level transition that is smoothed or a horizontal node that is smoothed (in which case smoothing some other horizontal nodes may be required).

The main technical issue to deal with is choosing the appropriate chain of degenerations so that the monodromy argument can be applied. Recall that the results of [4] apply to an rref basis of defining equations with respect to a chosen Γ–adapted homology basis. For an arbitrary chain of divisorial degenerations, it may happen that a homology basis that is Γ–adapted is not Γ_j–adapted for some $1 \le j < d$. We will thus choose the chain of divisorial degenerations so that a homology basis can be chosen that is Γ_j–adapted simultaneously for all $0 \le j \le d$. The chain of undegenerations that we choose is the following. Starting from $\Gamma_0 = \mathrm{pt}$, we first take degenerations that pinch horizontal nodes of top level in Γ, i.e. we take some $e_1 \in E^{\mathrm{hor}}_{(0)}(\Gamma)$ and take the divisorial degeneration $\Gamma_0 \rightsquigarrow \Gamma_1$ that pinches e_1 — which may possibly pinch some other horizontal nodes. If some top-level horizontal node is not yet pinched, we take $e_2 \in E^{\mathrm{hor}}_{(0)}(\Gamma) \setminus E(\Gamma_1)$, and take the divisorial degeneration $\Gamma_1 \rightsquigarrow \Gamma_2$ that pinches e_2, and continue in such a way until $E^{\mathrm{hor}}_{(0)}(\Gamma) = E(\Gamma_k)$. We then let Γ_{k+1} be the degeneration of Γ_k that pinches the level transition between levels 0 and -1, and then start pinching the horizontal nodes of level -1 in Γ until all of $E^{\mathrm{hor}}_{(-1)}(\Gamma)$ is pinched, then pinch the level transition between levels -1 and -2, and so on. We will want to track levels when we do this to eventually deal with a Γ–adapted basis. We now choose a chain $\mathrm{pt} = \Gamma_0 \rightsquigarrow \Gamma_1 \rightsquigarrow \cdots \rightsquigarrow \Gamma_d = \Gamma$ of undegenerations satisfying all the conditions mentioned above and fix it for the rest of the proof.

The first issue is that, a priori, requiring some horizontal node to be pinched may lead to some other horizontal nodes at a different level being pinched. We prove that this is not the case. (In order to avoid confusion, in this proof the top level of a cycle always refers to the level with respect to Γ and *not* with respect to some intermediate undegeneration).

Claim *If $\Gamma_i \rightsquigarrow \Gamma_{i+1}$ is a purely horizontal divisorial degeneration appearing in the chain of degenerations $\mathrm{pt} = \Gamma_0 \rightsquigarrow \Gamma_1 \rightsquigarrow \cdots \rightsquigarrow \Gamma_d = \Gamma$ constructed above, then all horizontal nodes that are pinched in the degeneration $\Gamma_i \rightsquigarrow \Gamma_{i+1}$ have the same top level (in Γ).*

Proof Let $H(i, i+1)$ be the collection of horizontal nodes pinched in the degeneration $\Gamma_i \rightsquigarrow \Gamma_{i+1}$. By the discussion after the statement of Theorem 1.5, we know that for any pair of nodes in $H(i,i+1)$ there exists a defining relation crossing exactly this pair of nodes. We then apply the monodromy argument (Proposition 3.1) to $\partial M_{\Gamma_i} \subseteq D_{\Gamma_i}$ and conclude that the vanishing cycles of nodes in $H(i,i+1)$ are pairwise proportional on ∂M_{Γ_i}. By taking the limit of this proportionality relation to the boundary point $p_0 \in \partial M_\Gamma$, we see that both horizontal nodes are of the same level in Γ, since otherwise the period over the vanishing cycle of one of the horizontal nodes would become zero, which is impossible. This proves the claim. □

We now claim that there exists a Γ–adapted homology basis that is Γ_j–adapted for all $0 \le j \le d$. Indeed, we first choose a Γ–adapted basis where the elements are ordered by level, with homology classes of higher top level appearing first. We then reorder the set of basis elements crossing horizontal nodes so that they appear in the order in which the corresponding horizontal nodes are pinched in our chain of degenerations. That is, if $j < i$ then any cycle of the Γ–adapted basis crossing a node in $E^{\mathrm{hor}}(\Gamma_j)$ is listed before any cycle crossing a node in $E^{\mathrm{hor}}(\Gamma_i) \setminus E^{\mathrm{hor}}(\Gamma_j)$. Note that because of the claim proved above the reordering only changes the position of cycles of the same level, so that cycles of higher levels still appear first. The resulting ordered Γ–adapted basis has the desired property that it is Γ_j–adapted for every j. We then choose this ordered Γ–adapted homology basis, and take the rref basis of defining equations with respect to it, which will thus be an rref basis for every Γ_j at once.

Since, by Lemma 3.7, M–cross-equivalence and rref-cross-equivalence are the same, it suffices to show that, for any equation F that is an element of the rref basis, the vanishing cycles of all horizontal nodes in $E^{\mathrm{hor}}(F)$ have proportional periods on M. Moreover, our proof will in fact give a certain rationality result: for any two nodes $e, e' \in E^{\mathrm{hor}}(F)$, there exist nonzero integers $n(e,e')$ and $n'(e,e')$ such that

$$n(e,e')\langle F, \lambda_e\rangle \int_{\lambda_e} \omega = n'(e,e')\langle F, \lambda_{e'}\rangle \int_{\lambda_{e'}} \omega \tag{3-5}$$

holds locally on M near p. Let H_i be the subset of $E^{\mathrm{hor}}(F)$ pinched at Γ_i, i.e. $H_i := E^{\mathrm{hor}}(F) \cap E(\Gamma_i)$.

We will prove by induction on i that the periods of ω over all the vanishing cycles corresponding to nodes in H_i are pairwise proportional on M. Since $H_d = E^{\mathrm{hor}}(F)$, the theorem will then follow. For the base case, we take the smallest $j > 0$ such that $H_j \ne \varnothing$. Applying the monodromy argument (Proposition 3.1) to the equations lost at the degeneration $\Gamma_{j-1} \rightsquigarrow \Gamma_j$ shows then that, for any two nodes in H_j, the periods over the corresponding vanishing cycles are proportional on $\partial M_{\Gamma_{j-1}}$. Denote by G this proportionality, considered as one of the defining linear equations of $\partial M_{\Gamma_{j-1}}$. Note that G is completely contained in the bottom level, since it is a relation between vanishing cycles contained in the bottom level. Thus, by Remark 2.2, we can lift the linear equation G for $\partial M_{\Gamma_{j-1}}$ to a local defining equation G' for M. A priori, G' could have additional terms of lower levels that disappear when restricting to $\partial M_{\Gamma_{j-1}}$. However, by construction of the chain of degenerations and by our choice of j, $\top(F) = \top(G) = -L(\Gamma_{j-1})$ is equal to the bottom level of the graph Γ_{j-1}. Thus, there are simply no levels below that level, and thus G' cannot have any additional summands at lower level. Thus, it follows that G' is a proportionality on M of these two nodes in H_j, which is thus one of the defining equations of M. This concludes the proof of the base case of the induction.

For the inductive step, we will prove the proportionality of periods of ω over the vanishing cycles for any two nodes in H_{i+1}, assuming that this holds for any pair of nodes in H_i. This statement is vacuously true if $H_i = H_{i+1}$, so the inductive step holds automatically unless $i \geq 1$ and $\Gamma_i \rightsquigarrow \Gamma_{i+1}$ is a purely horizontal degeneration. For such an undegeneration, denote by $\{h_1, \ldots, h_k\} := E(\Gamma_{i+1}) \setminus E(\Gamma_i)$ the new nodes that are pinched, all of which are horizontal. Since the degeneration $\Gamma_i \rightsquigarrow \Gamma_{i+1}$ is divisorial, the number of defining equations of M lost on Γ_{i+1} in addition to those lost on Γ_i must be equal to $k-1$, one less than the number of horizontal nodes pinched. Each of the lost equations has a pivot variable in the reduced row echelon form, and by Theorem 1.5 there exists exactly one horizontal node among the $h_1, \ldots, h_k$ (which by renumbering the h_j we will assume to be h_k) that is not a pivot for any of the lost equations. Then $h_1, \ldots, h_{k-1}$ all correspond to pivots of the lost equations, and, since all the equations are in reduced row echelon form, we see that F cannot cross any of these, since F was lost at the base case degeneration. On the other hand, $H_i \subsetneq H_{i+1}$ means that $E^{\mathrm{hor}}(F)$ must contain some h_j, and thus we must have $h_k \in E^{\mathrm{hor}}(F)$, and $H_{i+1} = H_i \sqcup \{h_k\}$.

We now apply the monodromy argument to F with respect to the boundary stratum $D_{\Gamma_{i+1}}$ considered within the closure boundary stratum $\overline{D}_{\Gamma_{j-1}}$, where j was the index introduced in the base case. This is to say, we consider monodromy around the vanishing cycles for the nodes in $E(\Gamma_{i+1}) \setminus E(\Gamma_{j-1})$, obtaining in this way a defining equation G of $\partial M_{\Gamma_{j-1}}$ that is a linear combination of periods over the vanishing cycles in $H_i \sqcup \{h_k\}$ and possibly also periods over some vertical vanishing cycles crossed by F, with all the coefficients nonzero. Note that all these vanishing cycles are of level $-L(\Gamma_{j-1})$ when considered in Γ_{j-1} and thus the equation G can be lifted to a linear equation on M, which by abuse of notation we will denote by G again. First consider the case where all the degenerations $\Gamma_{j-1} \rightsquigarrow \cdots \rightsquigarrow \Gamma_{i+1}$ are purely horizontal. In this case Γ_{i+1} has the same number of levels as Γ_{j-1} and F does not cross any vertical vanishing cycles. By induction, we already know that the periods of ω over the vanishing cycles of nodes in H_i are pairwise proportional on M, and substituting this into G implies that the period of ω over the remaining vanishing period λ_{h_k} is also proportional to these, on ∂M_{Γ_i}. The coefficient of $\int_{\lambda_{h_k}} \omega$ in G is given by monodromy, and is thus nonzero, so the period $\int_{\lambda_{h_k}} \omega$ is a nonzero multiple of the period over the vanishing cycle for any node in H_i. Proceeding exactly as in the base case of induction, we conclude that this proportionality is actually satisfied on M, and not only on ∂M_{Γ_i}. Tracing through the argument, we see that the proportionality relations are of the form (3-5).

It remains to treat the case where some undegeneration in the chain $\Gamma_{j-1} \rightsquigarrow \cdots \rightsquigarrow \Gamma_{i+1}$ is vertical. In this case the application of the monodromy argument to F might pick up additional contributions from the vertical vanishing cycles that F crosses. Note that the contributions from vertical vanishing cycles are (up to multiplication by a constant) independent of the order in which undegenerations are performed and which boundary point is chosen to apply the monodromy argument; this follows from the computation leading to (3-4) and is incorrect for horizontal vanishing cycles. Thus, to see that the contributions from vertical vanishing cycles vanish we can, starting from Γ_{j-1}, only perform vertical degenerations and then the contributions vanish by (3-4). □

Corollary 3.9 *Any pair of M–cross-related nodes lies on one level.*

Proof Let $e, e' \in E^{\mathrm{hor}}(\Gamma)$ be a pair of M–cross-related nodes, so that, by Theorem 1.1, the periods over the corresponding vanishing cycles are proportional on M. The rescaled limits of these periods are the (nonzero) residues of the twisted differential at the corresponding nodes. If one node is lower than the other, by definition of a multiscale differential compatible with a level graph this means that the limit of the ratio of these residues must be equal to zero. Since the residues are proportional with a constant coefficient, this means that both residues must be identically zero, which is impossible, as the multiscale differential must have simple poles at all horizontal nodes by definition. □

We now investigate defining equations that are non-top-horizontal-crossing equations. We recall that an equation F is called non-top-horizontal-crossing if it does not cross any horizontal nodes of level $\top(F)$. This allows the possibility that F might cross horizontal nodes of levels below $\top(F)$.

Lemma 3.10 *Let F be a defining equation of M that does not cross any horizontal nodes at level $\top(F)$, but crosses some horizontal node $e \in E^{\mathrm{hor}}_{(i)}(\Gamma)$ at level $i < \top(F)$. Then F can be written as the sum $F = H + G$ of defining equations such that $\top(G) = \top(F) > \top(H)$, where G crosses no horizontal nodes at any level, and H is top-horizontal-crossing with $\top(H)$ being the maximal level of a horizontal node crossed by F.*

Proof Using Remark 3.4, we construct a chain

$$\mathrm{pt} = \Gamma'_0 \rightsquigarrow \cdots \rightsquigarrow \Gamma'_k \rightsquigarrow \Gamma' \rightsquigarrow \Gamma.$$

as follows.

Here each $\Gamma'_i \rightsquigarrow \Gamma'_{i+1}$ is a divisorial degeneration pinching some horizontal node in $E^{\mathrm{hor}}(F)$, and we perform such divisorial degenerations until all nodes in $E^{\mathrm{hor}}(F)$ are pinched, i.e. $E(\Gamma'_k) = E^{\mathrm{hor}}(\Gamma)$. Then $\Gamma'_k \rightsquigarrow \Gamma'$ is the purely vertical degeneration that closes the level passage between $\top(F)$ and $\top(F) - 1$. Finally, $\Gamma' \rightsquigarrow \Gamma$ is the remaining degeneration, which closes all other level passages and all other horizontal nodes of Γ.

Then F is top-horizontal-crossing for all Γ'_j but not for Γ'. Since every defining equation for $\partial M_{\Gamma'}$ is induced by an equation of $\partial M_{\Gamma'_k}$, and every defining equation of $\partial M_{\Gamma'_{j+1}}$ is induced by an equation of $\partial M_{\Gamma'_j}$, it follows that each defining equation of $\partial M_{\Gamma'}$ is induced from a defining equation of M at p that does not cross *any* horizontal nodes. Thus, we can find an equation G_0 with the same top-level restriction as F, i.e. $(G_0)_\top = F_\top$, but such that G_0 crosses *no* horizontal nodes. In particular, then, $\top(F - G_0) < \top(F)$. Now either $F - G_0$ is top-horizontal-crossing, or we can proceed inductively and find G as desired. □

We can now prove the decomposition of the linear equations.

Proof of Theorem 1.4 We proceed by induction on $\#E^{\mathrm{hor}}(F) + \top(F)$. If F crosses no horizontal nodes, we set $G := F$ and are done. Otherwise, if all nodes in $E^{\mathrm{hor}}(F)$ are at levels strictly below $\top(F)$, we write $F = H + G$ as provided by Lemma 3.10, and apply the induction hypothesis on H.

The remaining case is that F crosses some horizontal node $e \in E^{\mathrm{hor}}(F) \cap E^{\mathrm{hor}}_{(\top(F))}(\Gamma)$. Given any defining equation of M, by the definition of primitivity, there exists some primitive equation P that crosses a subset of the horizontal nodes crossed by the original equation, i.e. $E^{\mathrm{hor}}(P) \subseteq E^{\mathrm{hor}}(F)$ and $\top(P) \geq \top(F)$. By Lemma 3.10, we can further assume that $\top(P) = \top(F)$. Then there exists a constant $c \in \mathbb{C}^*$ such that either $F = cP$ or $E^{\mathrm{hor}}(F - cP) \subsetneq E^{\mathrm{hor}}(F)$. We can then apply the induction to $F - cP$, and, since $E^{\mathrm{hor}}(P), E^{\mathrm{hor}}(F - cP) \subseteq E^{\mathrm{hor}}(F)$, condition (2) of the statement of the theorem will be satisfied. □

While all the above statements were for arbitrary defining equations, for the rref basis $(F_1, \dots, F_m)$ we can obtain more precise results, determining the coefficients of the equations explicitly. While this, our most precise, result, is more technical, it will be crucial in enabling us to compute the analytic equations of M in plumbing coordinates in Section 4, and in particular prove Theorem 1.7.

Proposition 3.11 *Let $F_1, \dots, F_m$ be the rref basis, written as in (2-3). Then:*

(1) *Each F_l does not cross any horizontal node at level below $\top(F_l)$.*

(2) *If F_l crosses $e, e' \in E^{\mathrm{hor}}_{(\top(F))}(\Gamma)$, then there exist two nonzero integers $n_1, n_2 \in \mathbb{Z}$ such that the equation*
$$n_1 \langle F_l, \lambda_e \rangle \int_{\lambda_e} \omega = n_2 \langle F_l, \lambda_{e'} \rangle \int_{\lambda_{e'}} \omega$$
holds on M in a neighborhood of p.

(3) *For any level i, the equation*
$$R_i(F_l) = \sum_{\substack{e \\ \ell(e^+) \geq i > \ell(e^-)}} m_{e,i} \langle F_l, \lambda_e \rangle \int_{\lambda_e} \omega = 0$$
holds on M in a neighborhood of p, where the $m_{e,i}$ are as defined in (2-1).

Proof We first prove (1). If F_l crosses any horizontal nodes of level below $\top(F)$, then consider the decomposition $F_l = H_1 + \dots + H_k + G$ provided by Theorem 1.4. After reordering the H_i we can assume that $\top(H_1) < \top(F)$. Writing $H_1 = \sum_k a_k F_k$, it follows then that $\top(F_k) < \top(F_l)$ whenever $a_k \neq 0$. Furthermore, there must exist j such that $a_j \neq 0$ and F_j crosses some horizontal nodes. Let e be the horizontal node corresponding to the pivot of F_j. Then $e \in E^{\mathrm{hor}}(H_1) \subseteq E^{\mathrm{hor}}(F_l)$, which is a contradiction since the pivot node can only appear in F_j, and in no other equation of the rref basis.

The proof of (2) was the content of (3-5), while the statement (3) was proved already in the proof of Theorem 1.2. □

4 Equations near the boundary in plumbing coordinates

Using the restrictions on linear equations obtained in Proposition 3.11, we can now convert the linear equations in period coordinates into analytic equations in plumbing coordinates and thus prove Theorem 1.6. The most precise technical result that we prove in this direction is Proposition 4.3.

4.1 Converting equations from period to plumbing coordinates

While periods of the differential are not globally well defined on U, recall that in [4] the so-called *log periods* were defined (these are related to perturbed periods of [3], and to the expressions for periods used in [10, Lemma 3.8]). These are well-defined analytic functions on U, obtained by subtracting logarithmic terms, as we now recall. As always, we work in the neighborhood U of $p_0 \in \partial M$, and consider defining equations of M at a smooth point $p = (X, \omega) \in M \cap U$; now we will also fix a class $\gamma \in H_1(X \setminus \underline{p}, \underline{z}; \mathbb{Z})$. Recall that coordinates on U are given by $b := (\eta, \underline{t}, \underline{h})$, where $\eta \in D_\Gamma$ can be thought of as a twisted differential, and thus local coordinates for η are given by its periods, $\underline{t} = \{t_{-1}, \dots, t_{-L(\Gamma)}\}$ are the level scaling parameters, and $\underline{h} = \{h_e\}_{e \in E^{\mathrm{hor}}(\Gamma)}$ are the plumbing parameters at horizontal nodes. The *log period* of ω along γ is defined as

$$\Psi_\gamma(\omega) := \frac{1}{t_{\lceil \top(\gamma) \rceil}} \left[\int_\gamma \omega - \sum_{e \in E} \langle \gamma, \lambda_e \rangle r_e(\omega) \ln(s_e) \right],$$

where $r_e(\omega) := (1/2\pi i) \int_{\lambda_e} \omega$ and, as in [3], we let

$$t_{\lceil i \rceil} := \prod_{k=-i}^{-1} t_i^{a_i},$$

with the a_i defined by (2-1). Here γ is extended smoothly to nearby curves using the Gauss–Manin connection. A priori, this might not be well defined because of the monodromy of the Gauss–Manin connection, but the logarithmic terms are chosen exactly to cancel out this monodromy, which yields:

Proposition 4.1 [4, Theorem 5.2] *The log period Ψ_γ is a well-defined analytic function on U. Furthermore, if γ is nonhorizontal, then*

$$\Psi_\gamma(b) = \int_\gamma \eta + H(b),$$

where H is an analytic function on U that vanishes identically on $D_\Gamma \cap U \subset U$.

We will now rewrite the defining equations of M at p in terms of log periods, and then express them in plumbing coordinates. We briefly recall our setup for writing in linear equations.

Let $(F_1, \dots, F_m)$ be the rref basis of defining equations for M, with respect to a fixed Γ–adapted basis.

To lighten the notation we focus on one of the equations $F := F_k$ for now.

Then we can write

$$F(X,\omega) = \sum_{i=-L(\Gamma)}^{\top(F)} \left(\sum_{l=1}^{c(i)} A_l^{(i)} \int_{\delta_l^{(i)}} \omega + \sum_{l=1}^{d(i)} B_l^{(i)} \int_{\alpha_l^{(i)}} \omega \right)$$

as in (2-4). Here we recall that $\delta^{(i)}$ are the top-horizontal-crossing cycles of level i and $\alpha^{(i)}$ are the non-top-horizontal-crossing cycles of level i for a fixed Γ–adapted basis.

By Proposition 3.11(1) all coefficients $A_l^{(j)}$ in (2-4) are zero for $j < \top(F)$. We thus write $i := \top(F)$ from now on, and $A_l := A_l^{(i)}$. In terms of log periods, we compute

$$\begin{aligned} F &= \sum_{l=1}^{c(i)} A_l \Big(t_{\lceil i \rceil} \Psi_{\delta_l^{(i)}} + \sum_{e\in E} \langle \delta_l^{(i)}, \lambda_e \rangle \cdot \int_{\lambda_e} \omega \cdot \ln(s_e) \Big) \\ &\qquad + \sum_{j=-L(\Gamma)}^{i} \sum_{l=1}^{d(j)} B_l^{(j)} \Big(t_{\lceil j \rceil} \Psi_{\alpha_l^{(j)}} + \sum_{e\in E} \langle \alpha_l^{(j)}, \lambda_e \rangle \cdot \int_{\lambda_e} \omega \cdot \ln(s_e) \Big) \\ &= \sum_{l=1}^{c(i)} A_l t_{\lceil i \rceil} \Psi_{\delta_l^{(i)}} + \sum_{j=-L(\Gamma)}^{i} \sum_{l=1}^{d(j)} B_l^{(j)} t_{\lceil j \rceil} \Psi_{\alpha_l^{(j)}} + \sum_{l=1}^{c(i)} A_l \cdot \int_{\lambda_l^{(i)}} \omega \cdot \ln(s_l^{(i)}), \end{aligned}$$

where we used Proposition 3.11(3) to substitute $R_j(F) = 0$ to obtain the cancellation of terms for the second equality. For future use, write

$$L(b) := \sum_{l=1}^{c(i)} A_l t_{\lceil i \rceil} \Psi_{\delta_l^{(i)}}(b) + \sum_{j=-L(\Gamma)}^{i} \sum_{l=1}^{d(j)} B_l^{(j)} t_{\lceil j \rceil} \Psi_{\alpha_l^{(j)}}(b).$$

If F is a non-top-horizontal-crossing equation, then all coefficients A_l are zero and we define

$$G(b) := \frac{1}{t_{\lceil \top(F) \rceil}} L(b) = \frac{1}{t_{\lceil \top(F) \rceil}} F(b), \tag{4-1}$$

which is then a holomorphic function on U.

If F is a top-horizontal-crossing cycle, then, by Theorem 1.1, the periods over vanishing cycles for all nodes in $E^{\mathrm{hor}}(F)$ are proportional. Since F is an element of the rref basis, we have $A_1 = 1$. Since the coefficients of proportionality determined explicitly in (3-5) are rational, it follows that there exist numbers $q_l \in \mathbb{Q} \setminus \{0\}$ such that

$$A_l \cdot \int_{\lambda_l^{(i)}} \omega = q_l \cdot \int_{\lambda_1^{(i)}} \omega.$$

We can thus write

$$F(b) = L(b) + \left(\int_{\lambda_1^{(i)}} \omega \right) \cdot \left(\ln(s_1^{(i)}) + \sum_{l=2}^{c(i)} q_l \ln(s_l^{(i)}) \right)$$

By clearing denominators (this is where the rationality of q_l is used), the equation $F(b) = 0$ is then equivalent to

$$\frac{n_1}{\int_{\lambda_1^{(i)}} \omega} L(b) + \sum_{l=1}^{c(i)} n_l \ln(s_l^{(i)}) = 0$$

for some nonzero integers n_l, and without loss of generality we can assume $\gcd(n_l) = 1$. By exponentiating, this is in turn equivalent to

$$e^{f(b)} \prod_{l=1}^{c(i)} (s_l^{(i)})^{n_l} = 1, \tag{4-2}$$

where we set

$$f(b) := \frac{n_1}{\int_{\lambda_1^{(i)}} \omega} L.$$

Since the point p_0 for which all $s_l^{(i)} = 0$ is contained in ∂M, we must have $f(0,0,0) = 0$. Thus, it cannot happen that all n_i have the same sign. By separating terms with n_l positive and negative, we can rewrite (4-2) as

$$0 = H(b) := e^{f(b)} s^I - s^J \tag{4-3}$$

for some monomials s^I and s^J in the plumbing parameters $s_l^{(i)}$. We have now converted an element F of the rref basis for defining equations of M at p to plumbing coordinates and can now repeat the same process for all remaining equations in the rref basis. Before proceeding with the general setup, we give an example of how this works in practice.

Example 4.2 Consider a boundary point p_0 such that the corresponding stable curve is irreducible — and in particular the level graph has one vertex and some horizontal edges. Let λ_1 and λ_2 be two horizontal vanishing cycles, and let δ_1 and δ_2 be crossing curves

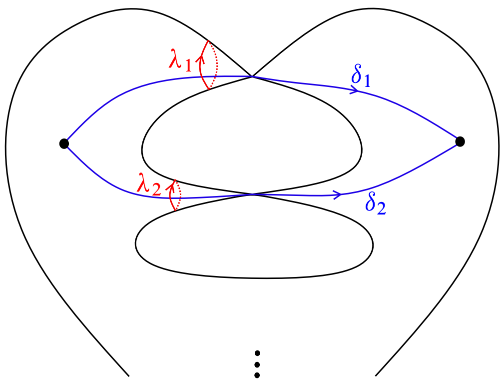


Figure 4: A boundary point with two horizontal node vanishing cycles and two curves crossing these.

for them, as shown in Figure 4. Suppose that M is locally cut out by the two equations in period coordinates given in the table below, where we note that Theorem 1.1 applied to the first equation implies an equation relating the two periods in the second — so that our machinery so far does not automatically produce any further equations. We will now demonstrate the procedure to convert these into equations in plumbing coordinates $b = (\eta, \underline{h})$ (since Γ has only one level, there are no scaling parameters $\underline{t}$). The result is given in the table below:

period equations	plumbing equations
$\int_{\delta_1} \omega - \int_{\delta_2} \omega = 0$	$e^f s_1 - s_2 = 0$
$\int_{\lambda_1} \omega - \int_{\lambda_2} \omega = 0$	$\int_{\lambda_1} \omega - \int_{\lambda_2} \omega = 0$

To convert the first period equation, we first express the first equation in terms of log periods:

$$\left(\Psi_{\delta_1} + \int_{\lambda_1} \omega \cdot \ln s_1\right) - \left(\Psi_{\delta_2} + \int_{\lambda_2} \omega \cdot \ln s_2\right) = 0.$$

Substituting here the second period equation yields

$$\Psi_{\delta_1} - \Psi_{\delta_2} + \left(\int_{\lambda_1} \omega\right)(\ln s_1 - \ln s_2) = 0.$$

Next we divide through and set $L = \Psi_{\delta_1} - \Psi_{\delta_2}$, which gives

$$\frac{L}{\int_{\lambda_1} \omega} + (\ln s_1 - \ln s_2) = 0.$$

Exponentiating and setting $f = L/\int_{\lambda_1} \omega$, we finally arrive at

$$e^f s_1 s_2^{-1} = 1,$$

so the first equation becomes $e^f s_1 - s_2 = 0$ in plumbing coordinates, as claimed.

The second period equation extends holomorphically to the boundary. Indeed, it does not involve any top-horizontal-crossing cycles, and thus no manipulations are necessary. ◁

4.2 Rearranging equations

Going back to the general case, suppose that among the rref basis $\{F_1, \dots, F_m\}$, there are u equations that are nonhorizontal, and $u' = m - u$ that are top-horizontal-crossing equations. We denote by $G_1, \dots, G_u$ the results of converting the non-top-horizontal-crossing equations to plumbing coordinates according to (4-1), and denote by $H_1, \dots, H_{u'}$ the results of converting the top-horizontal-crossing equations to plumbing coordinates according to (4-3). These can be then written as

$$G_k(b) = \frac{1}{t_{\lceil \top(F_k) \rceil}} L_k(b), \quad H_k(b) = e^{f_k(b)} s^{I_k} - s^{J_k} \tag{4-4}$$

for some monomials s^{I_k} and s^{J_k} in the variables $s_l^{(\top(F_k))}$. Note that, as $\underline{t}$ and $\underline{h}$ tend to zero, the equations G_k tend to the defining linear equations for ∂M_Γ.

If we now define

$$V := \{b \in U : G_1(b) = \cdots = G_u(b) = H_1(b) = \cdots = H_{u'}(b) = 0\},$$

then the defining linear equations of M can be rewritten analytically in plumbing coordinates as the equations defining V in plumbing coordinates.

We have thus proven:

Proposition 4.3 *The local irreducible component $\overline{Z}$ of $\overline{M}$ at p_0 containing p is a local irreducible component of V.*

We have thus proven a big part of Theorem 1.6: we have converted the defining equations into plumbing coordinates, and have given their explicit form in (4-4). The rest of the proof is a matter of organizing the equations.

Proof of Theorem 1.6 We now rearrange the equations to reveal some of the underlying structure. Let $l(1), \dots, l(u')$ be the pivots of those equations $F_{j_1}, \dots, F_{j_{u'}}$ that are top-horizontal-crossing. After a change of coordinates

$$x_l^{(i)} := \begin{cases} e^{f_{j_k}(b)} s_{l(k)}^{(\top(F_{j_k}))} & \text{if } l = l(k) \text{ and } i = \top(F_{j_k}), \\ s_l^{(i)} & \text{otherwise,} \end{cases}$$

the equations H_k take the form

$$H_k = x^{I_k} - x^{J_k}, \tag{4-5}$$

where I_k and J_k are the monomials from (4-4).

After this change of coordinates, we write the coordinates on U as $(\underline{y}, \underline{x})$, where $\underline{y}$ are all coordinates not involving horizontal nodes, and $\underline{x}$ is the set of plumbing parameters at the horizontal nodes that we just defined. Note that $\underline{y}$ can be separated further into the rescaling parameters $\underline{t}$ and the periods $\int_{\alpha_l^{(j)}} \eta$. We furthermore separate $\underline{x}$ into sets of coordinates corresponding to individual M–cross-equivalence classes, writing $\underline{x} = (\underline{x}_1, \dots, \underline{x}_a)$. In these coordinates the local irreducible component $\overline{Z}$ of $\overline{M} \cap U$ containing p is an irreducible component of the product

$$V = \{\underline{y} : G_1(\underline{y}) = \cdots = G_u(\underline{y}) = 0\} \times \prod_{l=1}^{a} \{\underline{x}_l : \underline{H}_l(\underline{x}_l) = (0, \dots, 0)\}, \tag{4-6}$$

where each $\underline{H}_l$ is the vector of all equations H_k crossing nodes in the M–cross-equivalence class $\underline{x}_l$; this is possible since all nodes crossed by H_k lie in the same M–cross-equivalence class.

We now show, that locally in the analytic topology, $\overline{Z}$ is isomorphic to a product of $\mathbb{C}^n$ and varieties defined by binomial equations. Since in the coordinates given by $(\underline{y}, \underline{x})$ each equation H_k is a difference of two monomials, it remains to show that the factor $\{\underline{y} : G_1(\underline{y}) = \cdots = G_u(\underline{y}) = 0\}$ is smooth and thus locally isomorphic to $\mathbb{C}^n$. This follows in particular from the proof of Corollary 4.5.

To finish the proof of the theorem, we recall the relation between binomial equations and toric varieties. Recall that by definition a toric variety X contains an algebraic torus $(\mathbb{C}^*)^n$ as an open dense subset, so that the action of $(\mathbb{C}^*)^n$ extends to X (note that here we do not require X to be a normal variety). By [19, Lemma 1.1] (see also [20]), the zero locus of a binomial prime ideal in $\mathbb{C}[x_1, \dots, x_n]$ is an irreducible toric variety. The ideal generated by the equations for V is generated by binomials but in general is not a prime ideal. Using the special form (4-4) of the equations for V we will explicitly construct an embedding of $(\mathbb{C}^*)^n$ in Z and thus show that $\overline{Z}$ is locally a toric variety.

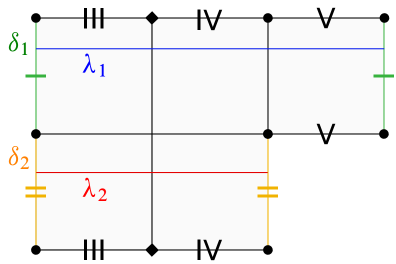


Figure 5: The square tiled surface from Example 4.4.

Since V, and hence also Z, is a product in $(\underline{y}, \underline{x})$–coordinates, it suffices to define the algebraic torus on each component cut out by $\underline{H}_l$. The crucial observation is that for each equation H_l the pivot variable only appears in H_l, and in no other equations. Thus, we define the $(\mathbb{C}^*)^n$–action explicitly in coordinates $z_1, \ldots, z_c$, where c is the number of nonpivots for $\underline{H}_l$, by sending z_k to x_k if x_k corresponds to a nonpivot, and for a pivot variable we define x_k as a function of $z_1, \ldots, z_c$ by solving the equation H_k for x_k. □

Example 4.4 The following example shows that the local irreducible component $\overline{Z}$ of the linear subvariety may not be normal, already for Teichmüller curves. Indeed, every 2–dimensional affine invariant submanifold contains *completely periodic surfaces* (X, ω), by which we mean that (X, ω) is a union of horizontal cylinders. Furthermore, for such (X, ω) the moduli of all horizontal cylinders on it are pairwise commensurable, by the Veech dichotomy (and also as easily follows from Theorem 1.5). Furthermore, all core curves of horizontal cylinders are pairwise proportional on M and there exists a choice of cross curves of the horizontal cylinders such that all cross curves are pairwise proportional as well. For two horizontal cylinders C_1 and C_2 on X, let e_1 and e_2 be the resulting horizontal nodes on the nodal curve obtained by applying $\left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$ to the horizontal cylinders, while fixing the rest of the surface. Then we can convert a linear relation $\int_{\delta_1} \omega = c \int_{\delta_2} \omega$ among the periods of core curves into the analytic equation

$$x^a = y^b,$$

where we let $a/b := m(C_1)/m(C_2)$, with $\gcd(a, b) = 1$, the ratio of the moduli of the two cylinders. For example, we consider the square-tiled surface from Figure 5 in the stratum $\Omega\mathcal{M}_{2,1}(2)$ with a horizontal cylinder of modulus 3 stacked on top of a horizontal cylinder of modulus 2. In this case the two linear relations defining M are

$$\int_{\delta_1} \omega = \int_{\delta_2} \omega, \quad 2 \int_{\lambda_1} \omega = 3 \int_{\lambda_2} \omega.$$

Converting them into analytic equations, we see that the local irreducible component $\overline{Z}$ is isomorphic to

$$\mathbb{C} \times \{(x, y) \mid x^2 = y^3\},$$

which is a product of $\mathbb{C}$ and a cusp, which is not normal, for example since the singular locus has codimension 1.

Corollary 4.5 *If all defining equations of M are nonhorizontal, then each local irreducible component $\overline{Z}$ of $\overline{M}$ is smooth and transverse to the vertical boundary stratum given by* $\{t_{-1} = \cdots = t_{-L(\Gamma)} = 0\}$.

Proof Indeed, in this case of only nonhorizontal equations we only have the first factor present in (4-6), and thus

$$V = \{\underline{y} : G_1(\underline{y}) = \cdots = G_u(\underline{y}) = 0\}.$$

For each equation G_k the pivot corresponds to a period $\int_{\alpha_{l(k)}^{(\top(G_k))}} \eta$, since G_k is nonhorizontal. By Proposition 4.1, the Jacobian of the set of equations $G_1, \ldots, G_u$ with respect to coordinates $\underline{y}$ is in reduced row echelon form and has the same pivots as the original linear equations $F_{j_1}, \ldots, F_{j_u}$ corresponding to nonhorizontal equations. In particular, V is smooth and irreducible, and, since it contains $\overline{Z}$, it must coincide with $\overline{Z}$. Furthermore, the normal space to V is generated by $\int_{\alpha_{l(k)}^{(\top(G_k))}} \eta$ for $k = 1, \ldots, u$ and thus we can choose $t_{-1}, \ldots, t_{-l(\Gamma)}$ as part of a local coordinate system on V, which shows that Z is transverse to $\{t_{-1} = \cdots = t_{-L(\Gamma)} = 0\}$. □

The condition of this corollary is satisfied for example if $\overline{M}$ is disjoint from the closed boundary divisor of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ that corresponds to graphs that have a horizontal edge. In Section 4.3 we will apply this corollary to obtain a compactification of Hurwitz spaces. We are now ready to prove our result about smoothing a collection of nodes of Γ.

Proof of Theorem 1.7 First note that the variety defined by equations $G_1(\underline{y}) = \cdots = G_u(\underline{y}) = 0$ is smooth and irreducible; we denote it by Y. As the local irreducible component $\overline{Z}$ of M is an irreducible component of V, which is a direct product, it follows that $\overline{Z}$ is a product of irreducible components of the factors, and we write it as $\overline{Z} = Y \times \prod_{l=1}^{a} X_l$, where the X_l denote the individual factors, which are given by equations in variables $\underline{x}_l$. As in the proof of Corollary 4.5 above, we know that there is a local coordinate system on Y including $(t_{-1}, \ldots, t_{-L(\Gamma)})$. Thus, for any sufficiently small collection of t_i, there exists a point in Y with these $\underline{t}$–coordinates, which is to say that any collection of level passages in Γ can be smoothed, while remaining in $\overline{M}$.

To show that any M–cross-equivalence class of horizontal nodes can be smoothed while remaining in $\overline{M}$, we simply observe that since $\overline{Z}$ is a product, and contains the flat surface $p \in \overline{Z} \cap M$, it means that the coordinates $\underline{x}_l(p)$ of this point are all nonzero, while $\underline{x}_l(p) \in X_l$. But then the point with all the same $\underline{y}$– and $\underline{x}$–coordinates as p_0, except with coordinates $\underline{x}_l(p)$, lies in the product $\overline{Z} = Y \times \prod_{l=1}^{a} X_l$, which is exactly to say the l^{th} M–cross-equivalence class of nodes has been smoothed. □

4.3 Application: a smooth compactification of Hurwitz spaces

As an application of our study of the local analytic equations of linear subvarieties, we construct a smooth compactification of Hurwitz spaces.

Recall that Hurwitz spaces are moduli spaces of rational functions on Riemann surfaces with prescribed ramification multiplicities. By associating to a rational function $f\colon X \to \mathbb{P}^1$ its exact differential df, we can consider Hurwitz spaces as subvarieties of meromorphic strata. Being an exact differential is characterized by the vanishing of all absolute periods, which are linear conditions in period coordinates. We can thus realize Hurwitz spaces as linear subvarieties of strata. The Hurwitz spaces we consider here are a "rigidified" version of the standard Hurwitz spaces where we mark all points lying over a branch point. If we only mark the points over two fibers, for example the fiber over 0 and ∞, then we arrive at the definition of double ramification cycles instead. In [5], the first author will use a similar approach to describe the closure of double ramification loci inside $\mathcal{M}_{g,n}$.

We now briefly define the Hurwitz spaces that we consider. Let $f\colon X \to \mathbb{P}^1$ be a degree d map, which we think of as a rational function, branched over $x_1, \dots, x_n \in \mathbb{P}^1$, with local ramification indices $(e_1^{(i)}, \dots, e_{k_i}^{(i)})$ over x_i. We call the tuple

$$\underline{d} = (d; (e_1^{(1)}, \dots, e_{i_1}^{(1)}), \dots, (e_1^{(n)}, \dots, e_{i_n}^{(n)}))$$

the *branching profile* of f. For a fixed branching profile $\underline{d}$, we define the *Hurwitz space*

$$\operatorname{Hur}(\underline{d}) := \Big\{(X, \underline{z}, f\colon X \to \mathbb{P}^1) \;\Big|\; f \text{ has branching profile } \underline{d},\ \operatorname{mult}_{z_k^{(i)}} f = e_k^{(i)} \text{ and } f(z_k^{(i)}) = f(z_{k'}^{(i)}) \text{ for all } k, k', \Big\} \Big/ \sim, \tag{4-7}$$

where $\underline{z} = (z_1^{(1)}, \dots, z_{i_1}^{(1)}, \dots, z_1^{(n)}, \dots, z_{i_n}^{(n)}) \subset X$ is a collection of distinct labeled points and $\operatorname{mult}_z f$ denotes the ramification index of f at z. Two such covers $(X, \underline{z}, f)$

and $(X', \underline{z}', f')$ are considered equivalent if there exists an isomorphism $\phi\colon (X, \underline{z}) \to (X', \underline{z}')$ of pointed Riemann surfaces and an isomorphism $\psi\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that the diagram

$$\begin{array}{ccc} (X, \underline{z}) & \xrightarrow{\ \phi\ } & (X', \underline{z}') \\ \big\downarrow f & & \big\downarrow f' \\ \mathbb{P}^1 & \xrightarrow{\ \psi\ } & \mathbb{P}^1 \end{array}$$

commutes. Note that, for every $(X, \underline{z}, f)$, after composition with an automorphism of $\mathbb{P}^1$, we can assume that $f^{-1}(\infty) = \{z_1^{(n)}, \dots, z_{i_n}^{(n)}\}$. After this normalization we can still translate and rescale f. Since df is unchanged when f is translated, a rational function up to automorphisms is the same as an exact differential up to rescaling.

Given a branching profile $\underline{d}$, we define a partition

$$\mu = (\mu_1^1, \dots, \mu_{i_1}^{(1)}, \dots, \mu_1^{(n)}, \dots, \mu_{i_n}^{(n)})$$

of $2g-2$ by setting $\mu_k^{(i)} := \mathrm{ord}_{z_k^{(i)}}\, df$, where we normalize as above, so that f is assumed to have poles exactly at $z_1^{(n)}, \dots, z_{i_n}^{(n)}$. Thus, thinking of the triple $(X, \underline{z}, df)$ instead of $(X, \underline{z}, f)$ gives a map of the Hurwitz space to the projectivized stratum $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$, and we thus see that $\mathrm{Hur}(\underline{d})$ is isomorphic to the linear subvariety

$$\mathbb{P}\Omega\mathrm{Hur}(\underline{d}) := \Big\{(X, \underline{z}, \omega) \in \mathbb{P}\mathcal{M}_{g,n}(\mu) \;\Big|\; \int_\gamma \omega = 0 \text{ for all } \gamma \in H_1(X; \mathbb{Z}), \\ \int_{p_k^{(i)}}^{p_{k'}^{(i)}} \omega = 0 \text{ for all } k, k', i \neq n\Big\}.$$

We can thus compactify $\mathrm{Hur}(\underline{d})$ by taking the closure of $\mathbb{P}\Omega\mathrm{Hur}(\underline{d})$ inside $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$.

Proposition 4.6 *The closure $\overline{\mathbb{P}\Omega\mathrm{Hur}(\underline{d})} \subseteq \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is a* ***smooth*** *compactification of $\mathbb{P}\Omega\mathrm{Hur}(\underline{d})$.*

Proof For any boundary point $p_0 \in \partial\mathbb{P}\Omega\mathrm{Hur}(\underline{d})$, we first claim that Γ has no horizontal nodes. Indeed, at a horizontal node, the residue of a twisted differential is nonzero, but since the twisted differential is the rescaled limit of exact differentials df, all of whose absolute periods are zero on all flat surfaces in $\mathbb{P}\Omega\mathrm{Hur}(\underline{d})$, this is impossible.

Thus, by Corollary 4.5, every local irreducible component of $\overline{\mathbb{P}\Omega\mathrm{Hur}(\underline{d})}$ at p_0 is smooth. Note that Corollary 4.5 has been stated only for unprojectivized strata but the proof applies also for projectivized strata, as the extra local factor of $\mathbb{C}^*$ does not make any difference.

It remains to show that $\overline{\mathbb{P}\Omega\mathrm{Hur}(\underline{d})}$ is locally irreducible at p_0. Assume that $\overline{Z}_1$ and $\overline{Z}_2$ are two local irreducible components of $\overline{\mathbb{P}\Omega\mathrm{Hur}(\underline{d})}$ near p_0, and choose smooth points $p_i = (X_i, \omega_i) \in Z_i$. Given a Γ–adapted basis at p_1 and a path γ from p_1 to p_2, we can transport it to a Γ–adapted basis at p_2 using the Gauss–Manin connection along γ. The resulting homology basis at p_2 depends on γ, but the bases at p_2 obtained by translating along different paths will only differ by adding multiples of the vanishing cycles. Since every vanishing cycle is contained in absolute homology, the resulting analytic equations for $\overline{Z}_2$ will be independent of the choice of γ. Let N be the defining linear equations for $\overline{Z}_1$ at p_1 in the chosen Γ–adapted basis, and let $\mathrm{GM}(N)$ be the result of transporting them along γ using the Gauss–Manin connection. Then $\mathrm{GM}(N)$ are defining equations for $\overline{Z}_2$ at p_2, simply because they are again the equations of vanishing of all absolute periods, and the vanishing of relative periods (which is a condition that is independent of the choice of the path, as all absolute periods are zero). Using (4-4), we can convert N and $\mathrm{GM}(N)$ into the analytic equations of $\overline{Z}_1$ and $\overline{Z}_2$ near p_0 in plumbing coordinates. Since N and $\mathrm{GM}(N)$ induce the same analytic equations near p_0, at every boundary point there can be only one local irreducible component, and it finally follows that the closure is smooth. □

5 Cylinder deformation theorem

In this section we use the restrictions on linear equations that we have obtained (in particular, Theorems 1.4 and 1.1) to give a new proof of the cylinder deformation theorem. The key point is that we can decompose the defining equations in such a way that all the cylinders crossed by any particular equation are M–parallel. It then follows that M admits some deformation changing just the cylinders in an M–parallel equivalence class, and in fact we show that stretching/shearing all these cylinders by the same matrix remains in M.

Below, we will need to consider a *cross-curve* δ of a cylinder C on a flat surface p_0, in the sense of Wright [21]. This is defined to be a curve represented by a saddle connection that lies in the cylinder, crosses the cylinder, and has one endpoint at a zero on the bottom boundary of the cylinder and the other endpoint at a zero on the top boundary (note that a cross-curve can cease to be a cross-curve under a small perturbation, for instance if the cylinder contains multiple zeroes on each of its boundary components).

We start with a lemma that gives a connection between horizontal nodes and Euclidean cylinders for the flat metric of large modulus.

Lemma 5.1 *For any $p_0 \in D_\Gamma \subset \partial\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, there exists a sufficiently small neighborhood $U \ni p_0$ and a sufficiently large $R > 0$ such that, for any flat surface $p = (X,\omega) \in U \cap \Omega\mathcal{M}_{g,n}(\mu)$ and for any flat Euclidean cylinder $C \subset X$ of modulus greater than R, the circumference curve λ of C is a horizontal vanishing cycle.*

Proof We first show that the core curve of every essential annulus of sufficiently large modulus must be homotopic to a vanishing cycle. While this is an easy standard argument, we have not been able to pinpoint a precise reference in the literature. For this, we forget the flat structure, and work in a neighborhood U of a nodal curve $X_0 \in \partial\overline{\mathcal{M}}_{g,n}$, where every smooth curve has a thick–thin decomposition, where we think of U in terms of standard plumbing coordinates near the boundary of $\overline{\mathcal{M}}_{g,n}$. Let λ be the core curve of an essential annulus on X of sufficiently large modulus R. Then, by the Schwarz lemma, the homotopy class of λ contains a short closed geodesic λ' for the hyperbolic metric on X (where short means of length going to zero as $R \to \infty$). We claim that λ' cannot intersect the thick part of X.

To this end, observe that the hyperbolic length of all closed geodesics on the thick part of all $X \in U$ is bounded below by a nonzero constant, and thus, by increasing R if necessary, we can ensure that it cannot happen that λ' is contained in the thick part. If λ' intersects both the thin and thick parts of X, consider a "shortened" plumbing annulus, where collars of hyperbolic width 1 are fixed at both ends. Then, by using this smaller plumbing neighborhood to start with, we can ensure that λ' must intersect both the thin part in the shortened plumbing annulus and the thick part for the original longer plumbing annulus. In particular, λ' must cross from one boundary of the collar to the other, but then the hyperbolic length of λ' must be at least 1, so λ' cannot be short. Thus, finally, λ' must be contained in the thin part, but then it must be contained in one plumbing annulus, and finally it must be homotopic to the corresponding vanishing cycle.

We now switch from this general discussion to the situation of essential annuli that are Euclidean cylinders for the flat metric. Choose a neighborhood U of $p_0 \in \Xi\overline{\mathcal{M}}_{g,n}(\mu)$ sufficiently small that every $p = (X,\omega) \in U$ is obtained by plumbing some $p' \in D_\Gamma$ under the plumbing construction of [3, Section 10], and so that the above argument applies for some chosen large R. We claim that (possibly after further increasing R and shrinking U) the core curve of any Euclidean cylinder C is homotopic to a horizontal vanishing cycle.

Suppose for contradiction that the core curve of C is homotopic to some vertical vanishing cycle λ_e with $e \in E^{\mathrm{ver}}(\Gamma)$. Recall that the plumbing construction for flat differentials glues in a plumbing annulus $\mathbb{V}_e$ around a vertical node such that ω on it has the standard form Ω_e given by [3, (10.8)]. In particular, ω has no zeroes or poles on $\mathbb{V}_e$. The cross-curve δ of C connects two zeroes of ω and thus must cross into the thick part of both $X_{(\ell(e-))}$ and $X_{(\ell(e+))}$. In particular, we can choose a geodesic λ' for the flat metric on C that is in the isotopy class of λ_e and passes through some fixed point x in the thick part of $X_{\ell(e+)}$. Let D be a small disk of fixed radius around x, contained in the thick part of X. Then the length $\left|\int_\lambda \omega\right| = \int_{\lambda'} |\omega|$ of λ' in the flat metric is bounded below by

$$\int_{\lambda' \cap D} |\omega| = c \cdot |t_{\lceil \ell(e+) \rceil}|,$$

where $t_{\lceil \ell(e+) \rceil}$ is the scaling parameter for ω on $X_{(\ell(e+))}$ and c is a constant independent of ω, which depends on the size of D and the choice of the thick–thin decompositions. Note that c depends on which zeros are connected by the cross-curve δ, but since there are only finitely many zeros we can choose c to be the minimum.

On the other hand, λ_e is homotopic to a path contained in the thick part of $X_{(\ell(e-))}$ and thus the length of λ_e can be bounded above by $c' \cdot |t_{\lceil \ell(e-) \rceil}|$ for some constant c' independent of ω. This contradicts the fact that $|t_{\lceil \ell(e+) \rceil}| \gg |t_{\lceil \ell(e-) \rceil}|$ on U (after possibly further shrinking U). □

We are now ready to prove our generalization of the cylinder deformation theorem.

Proof of Theorem 1.9 We begin by using the $\mathrm{GL}^+(2,\mathbb{R})$–action to get to a boundary point of M where we can apply our results restricting the defining equations of M. Recall that elements $a_t, u_s \in \mathrm{GL}^+(2,\mathbb{R})$ can be applied to any flat surface, and that a_t and u_s preserve M, since M is given by linear equations with real coefficients. Recall that transformations $a_t^{\mathcal{C}}$ and $u_s^{\mathcal{C}}$ only act on cylinders in the class $\mathcal{C}$, leaving the rest of the flat surface unchanged, and our goal is to show that they also preserve M.

For any given (X,ω), the forward orbit $\{a_t(X,\omega)\}_{t\geq 0}$ is contained in M. Since all cylinders in $\mathcal{C}$ are stretched unboundedly by a_t as $t \to +\infty$, the underlying Riemann surfaces in this orbit degenerate as $t \to +\infty$. Thus, the image of this forward orbit in the projectivization $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ cannot be compact, and there must exist a boundary point $\mathbb{P}p_0 = \mathbb{P}(X_0,\Gamma,\eta_0) \in \partial\mathbb{P}\,\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ and a sequence $\{t_n'\}$ of positive numbers such that $t_n' \to +\infty$ and $\mathbb{P}a_{t_n'}(X,\omega) \to \mathbb{P}p_0$. Here by $\mathbb{P}p_0$ we mean the image in $\mathbb{P}\,\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ of a point $p_0 \in \Xi\overline{\mathcal{M}}_{g,n}(\mu)$, as throughout the paper, under the quotient map $\Xi\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\,\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. This implies that there exist complex numbers r_n

such that $r_n a_{t'_n}(X,\omega) \to p_0$. By taking a subsequence, we can assume that the angles of the r_n converge to some $\alpha \in S^1$. We now rotate each r_n so that it is positive and real, and we replace p_0 with $-\alpha p_0$. In the end we get a sequence r_n of positive reals such that $r_n a_{t'_n}(X,\omega) \to p_0$.

By throwing away some of the beginning terms of the sequence, we can assume that all $r_n a_{t'_n}(X,\omega)$ lie in U (recall that U is a small neighborhood of the boundary point p_0). Let $(Y,\omega_Y) := a_{t'_1}(X,\omega)$. It suffices to prove the statement for (Y,ω_Y) instead of (X,ω).

We now subdivide U into a finite number of simply connected sets (as in Section 3.1 of [10] or Section 8 of [8]). By passing to a subsequence, we can assume that $r_n a_{t'_n} X$ all lie in one of these sets W. Let $\{t_n := t'_n - t'_1\}$ be the sequence such that $a_{t_n} Y = a_{t'_n} X$. Below we will not need to think of the 1–form separately from the Riemann surface, so we will drop ω_Y from the notation; we will denote by $\beta(Y) := \int_\beta \omega_Y$ the period over a relative homology class β.

By Lemma 5.1, the circumference curve of each horizontal cylinder C_i on $a_{t_n} Y$ must be the vanishing cycle for some horizontal node of p_0 (note that we are abusing notation by thinking of C_i, initially defined to be a cylinder on X, as a cylinder on $a_{t_n} Y$; this creates no issues since all of these surfaces are in the a_t–orbit of X). By passing to a further subsequence of t_n, we can assume that the horizontal node $e_i^{(n)} \in E^{\mathrm{hor}}(\Gamma)$ whose vanishing cycle is the circumference curve of C_i on $a_{t_n} Y$ does not in fact depend on n.

By Theorem 1.4, any defining equation F of M can be decomposed as

$$F = H_1 + \cdots + H_k + G,$$

where each H_j crosses a primitive collection of horizontal nodes, all at level $\top(H_j)$, and G does not cross any horizontal nodes. To show that $a_t^{\mathcal{C}} u_s^{\mathcal{C}} Y \in M$, it is thus enough to show that any such defining equations $H_1, \ldots, H_k, G$ vanish also at the point $a_t^{\mathcal{C}} u_s^{\mathcal{C}} Y$.

We will express this deformation in terms of periods of cross-curves (see the discussion at the beginning of this section for the definition). Let δ_i be a cross-curve of the cylinder C_i on the surface Y. Since C_i is horizontal, its height $h_i(Y)$ on the surface Y equals $\operatorname{Im} \delta_i(Y)$. Note that δ_i can be thought of as a relative homology class on all surfaces in the simply connected set W. For sufficiently small t and s, the deformation $a_t^{\mathcal{C}} u_s^{\mathcal{C}}$ changes periods by

(5-1) $$\delta_i(Y) \mapsto a_t u_s(\delta_i(Y))$$

for any i, while preserving the period over any curve that does not cross any cylinder C_i. Since the class G does not cross any horizontal nodes, and in particular does not cross the circumference curve of any C_i, it follows that $G(a_t^{\mathcal{C}} u_s^{\mathcal{C}} Y) = G(Y) = 0$ for any sufficiently small t and s.

For a defining equation H_j, first note that if it does not cross any of the cylinders in $\mathcal{C}$, then it is similarly preserved under the deformation $a_t^{\mathcal{C}} u_s^{\mathcal{C}}$. Suppose now that H_j crosses some $C_i \in \mathcal{C}$. Since the collection of horizontal nodes that H_j crosses is primitive, all of these nodes are M–cross-related. Hence, by Theorem 1.1, the periods over all the vanishing cycles crossed by H_j are proportional on M, and hence all the cylinders crossed by H_j are M–parallel. Since $\mathcal{C}$ is a full equivalence class of M–parallel cylinders, all of the cylinders crossed by H_j must lie in $\mathcal{C}$.

We can thus write

$$H_j = \beta_j + \sum_{i=1}^{d} c_{i,j}\delta_i, \tag{5-2}$$

where β_j is a relative homology class that does not cross any horizontal nodes, $\top(\beta_j) \le \top(H_j)$ and $c_{i,j}$ are some real numbers. Furthermore, $\top(\delta_i) = \top(H_j)$ for all i, since H_j has the same top level as the cylinders in $\mathcal{C}$, and δ_i is a cross-curve of such a cylinder.

Without the β_j term, $a_t^{\mathcal{C}} u_s^{\mathcal{C}}$ would act on H_j in exactly the same way that $a_t u_s$ does, and $H_j(a_t^{\mathcal{C}} u_s^{\mathcal{C}} Y) = 0$ would follow from the fact that M is defined by linear equations with real coefficients, which are preserved by the $\mathrm{GL}^+(2,\mathbb{R})$–action. The presence of the β_j term makes the proof more complicated. We will use our sequence $a_{t_n} Y$ to prove the following:

Claim *The imaginary part* $\operatorname{Im} \beta_j(Y)$ *is zero.*

Assuming the claim, the fact that $H_j(a_t^{\mathcal{C}} u_s^{\mathcal{C}} Y) = 0$ follows easily. Indeed, we first compute the difference

$$\begin{aligned} H_j(a_t^{\mathcal{C}} u_s^{\mathcal{C}} Y) &- a_t u_s H_j(Y) \\ &= \Big(\beta_j(a_t^{\mathcal{C}} u_s^{\mathcal{C}} Y) + \sum_i c_{i,j}\delta_i(a_t^{\mathcal{C}} u_s^{\mathcal{C}} Y)\Big) - \Big(a_t u_s \beta_j(Y) + a_t u_s \sum_i c_{i,j}\delta_i(Y)\Big) \\ &= \Big(\beta_j(Y) + a_t u_s \sum_i c_{i,j}\delta_i(Y)\Big) - \Big(a_t u_s \beta_j(Y) + a_t u_s \sum_i c_{i,j}\delta_i(Y)\Big) \\ &= \beta_j(Y) - a_t u_s(\beta_j(Y)) = 0, \end{aligned}$$

where in the last equality we used the claim: since $\operatorname{Im}\beta_j(Y)=0$, one computes $a_t u_s(\beta_j(Y))=\beta_j(Y)$. Now, since $H_j(Y)=0$, we have $a_t u_s H_j(Y)=0$, and hence the above implies $H_j(a_t^{\mathcal{C}}u_s^{\mathcal{C}}Y)=0$, as desired.

To complete the proof of the theorem, it thus remains to prove the claim, for which we will use the convergent sequence $r_n a_{t_n} Y\to p_0$ constructed in the beginning of the proof. Since $r_n a_{t_n} Y\in M$, we know that $H_j(r_n a_{t_n} Y)=0$. Taking the imaginary part and using the expression (5-2) gives

$$\operatorname{Im}(\beta_j(r_n a_{t_n} Y))+\sum_i c_{i,j}\operatorname{Im}(\delta_i(r_n a_{t_n} Y))=0 \tag{5-3}$$

(note that this again uses the fact that we are working with a linear variety defined by equations with real coefficients, so that $c_{i,j}$ are real).

The curve δ_i on $r_n a_{t_n} Y$ is a curve that crosses the cylinder C_i. While δ_i on $r_n a_{t_n} Y$ is not necessarily a cross-curve in the sense above, we claim that it has the same top level as the vanishing cycle of C_i. Indeed, to see this one argues as in the proof of Lemma 5.1: if δ_i had higher top level, then one could choose a closed geodesic representing the circumference curve of C_i that would cross the thick part of the surface at this higher level, which would then have length much larger than the magnitude of the period over the vanishing cycle; on the other hand, since δ_i crosses C_i, its top level is at least the level of C_i. Since C_i is a horizontal cylinder, its height $h_i(r_n a_{t_n} Y)=r_n e^{t_n} h_i(Y)=r_n e^{t_n}\operatorname{Im}\delta_i(Y)$ on the surface $r_n a_{t_n} Y$ is approximated by $\operatorname{Im}\delta_i(r_n a_{t_n} Y)$; in fact,

$$\operatorname{Im}\delta_i(r_n a_{t_n} Y)-r_n e^{t_n}\operatorname{Im}\delta_i(Y)=o(r_n e^{t_n})$$

as $n\to\infty$. Substituting this into (5-3) gives

$$\operatorname{Im}\beta_j(r_n a_{t_n} Y)+\sum_i c_{i,j}(r_n e^{t_n}\operatorname{Im}\delta_i(Y)+o(r_n e^{t_n}))=0,$$

and dividing through by $r_n e^{t_n}$ gives

$$\frac{\operatorname{Im}(\beta_j(r_n a_{t_n} Y))}{r_n e^{t_n}}+\sum_i c_{i,j}\operatorname{Im}\delta_i(Y)=0.$$

Recall that β_j is a curve with top level $\top(\beta_j)\le\top(H_j)$, while $\top(H_j)$ is the level of the circumference curve of each of the cylinders C_i. Hence, on surfaces in W, the magnitude of the period of β_j is less than a constant multiple of the circumference of C_i. The height of each C_i on $r_n a_{t_n} Y$ is within a constant factor of $r_n e^{t_n}$. Each cylinder C_i is degenerating as $n \to \infty$, so its modulus is going to infinity. It follows that the left-hand term in the above goes to 0 as $n \to \infty$. Taking the limit, we get

$$\sum_i c_{i,j} \operatorname{Im} \delta_i(Y) = 0.$$

Combining this with the fact that at Y the imaginary part of (5-2) is 0, we get $\operatorname{Im} \beta_j(Y)) = 0$, as claimed. □

6 The linear equations of affine invariant submanifolds

In this section we specialize our study of linear subvarieties to the case of affine invariant submanifolds. In our language, this is simply to say that we are talking about linear subvarieties of holomorphic strata (so all $m_i > 0$) such that furthermore all the defining linear equations have real coefficients. Avila, Eskin and Möller [1] show that, for any affine invariant manifold M in a holomorphic stratum, the image of the tangent space $T_{(X,\omega)}M \subset H^1(X, \underline{z}; \mathbb{C})$ in $H^1(X; \mathbb{C})$ is symplectic under the natural symplectic pairing. We first carefully set up notation for all this.

6.1 General setup

Denote by

$$\iota\colon H_1(X;\mathbb{C}) \hookrightarrow H_1(X,\underline{z};\mathbb{C}), \quad u\colon H_1(X \setminus \underline{z};\mathbb{C}) \twoheadrightarrow H_1(X;\mathbb{C})$$

the natural maps, and by abuse of notation denote by $\langle\, , \rangle$ both natural intersection pairings

$$H_1(X;\mathbb{C}) \times H_1(X;\mathbb{C}) \to \mathbb{C}, \quad H_1(X,\underline{z};\mathbb{C}) \times H_1(X \setminus \underline{z};\mathbb{C}) \to \mathbb{C},$$

which satisfy the adjunction property

$$\langle x, u(v)\rangle = \langle \iota(x), v\rangle$$

for any $x \in H_1(X;\mathbb{C})$ and $v \in H_1(X \setminus \underline{z};\mathbb{C})$. Given a subspace $V \subset H_1(X;\mathbb{C})$ (or of $H_1(X,\underline{z};\mathbb{C})$, respectively $H_1(X\setminus\underline{z};\mathbb{C})$), we denote by $V^\perp$ the perp space with respect to $\langle\, , \rangle$ in $H_1(X;\mathbb{C})$ (respectively of $H_1(X\setminus\underline{z};\mathbb{C})$ or $H_1(X,\underline{z};\mathbb{C})$). For a subspace V of homology, we denote by $\operatorname{Ann} V$ its annihilator in cohomology.

The following result controls the space of deformations in M supported on an equivalence class of M–parallel cylinders, modulo purely relative deformations. In the below we will study small deformations, which are elements of the tangent space $TM \subset H^1(X,\underline{z};\mathbb{C})$ at the given point (X,ω).

Lemma 6.1 *Let M be an affine invariant manifold and let $\mathcal{C}$ be an equivalence class of M–parallel cylinders on some $(X,\omega) \in M$. Let $V \subseteq H_1(X \setminus \underline{z};\mathbb{C})$ be the span of the circumference curves of the cylinders in $\mathcal{C}$. Then*

$$\dim \iota^*(TM \cap \operatorname{Ann} V^{\perp}) \leq 1.$$

Note that $V^{\perp}$ consists of all homology classes that don't intersect one of the cylinder circumference curves in $\mathcal{C}$, so $\operatorname{Ann} V^{\perp}$ is the space of local deformations in the stratum supported on the union of these cylinders. Hence, one can think of $\iota^*(TM \cap \operatorname{Ann} V^{\perp})$ as local deformations in M supported on the union of cylinders in $\mathcal{C}$, modulo purely relative deformations.

The proof of the lemma is a simple application of the result of Avila, Eskin and Möller [1] that $\iota^*(TM)$ is symplectic, together with some formal linear algebra.

Proof We first note that in our linear algebra setup, for any subspace $W \subseteq H_1(X \setminus \underline{z},\mathbb{C})$, we have

$$\iota(u(W)^{\perp}) \subseteq W^{\perp}, \tag{6-1}$$

while, for any subspace $Z \subseteq H_1(X;\mathbb{C})$, we have

$$\iota^*\big(\operatorname{Ann}(\iota(Z))\big) \subseteq \operatorname{Ann}(Z). \tag{6-2}$$

Using these two facts, we get

$$\begin{aligned} \iota^*(TM \cap \operatorname{Ann} V^{\perp}) &\subseteq \iota^*(TM) \cap \iota^*(\operatorname{Ann} V^{\perp}) \\ &\subseteq \iota^*(TM) \cap \iota^*\big(\operatorname{Ann}(\iota(u(V)^{\perp}))\big) \\ &\subseteq \iota^*(TM) \cap \operatorname{Ann}(u(V)^{\perp}). \end{aligned} \tag{6-3}$$

The above are subspaces of absolute cohomology, but the symplectic form is easier to understand in absolute homology, so we take the annihilator of the last term above:

$$\begin{aligned} \operatorname{Ann}\big(\iota^*(TM) \cap \operatorname{Ann}(u(V)^{\perp})\big) &= \operatorname{Ann}(\iota^*(TM)) + \operatorname{Ann}\big(\operatorname{Ann}(u(V)^{\perp})\big) \\ &= \operatorname{Ann}(\iota^*(TM)) + u(V)^{\perp}. \end{aligned}$$

By this equality and (6-3), to prove the lemma it suffices to show that

$$\dim\big(\operatorname{Ann}(\iota^*(TM)) + u(V)^{\perp}\big) \geq n-1, \tag{6-4}$$

where $n := \dim H_1(X;\mathbb{C})$.

Now recall that V is spanned by circumference curves of M–parallel cylinders, which is to say they remain parallel under small deformations in TM. Thus, annihilating them imposes only one condition on TM, which is to say we have

$$\dim(\mathrm{Ann}(\iota^* TM) \cap u(V)) = \dim u(v) - 1. \tag{6-5}$$

By [1], $\iota^*(TM) \subset H^1(X;\mathbb{C})$ is symplectic subspace, and hence so is $\mathrm{Ann}(\iota^*(TM))$. It follows that

$$\dim\big(u(V) \cap \mathrm{Ann}(\iota^*(TM))\big) + \dim\big(u(V)^\perp \cap \mathrm{Ann}(\iota^*(TM))\big) = \dim \mathrm{Ann}(\iota^*(TM)).$$

Combining this with (6-5) gives

$$\dim\big(u(V)^\perp \cap \mathrm{Ann}(\iota^*(TM))\big) = \dim \mathrm{Ann}(\iota^*(TM)) - \dim u(v) + 1. \tag{6-6}$$

Thus,

$$\begin{aligned}
&\dim\big(u(V)^\perp + \mathrm{Ann}(\iota^*(TM))\big) \\
&\qquad = \dim u(V)^\perp + \dim \mathrm{Ann}(\iota^*(TM)) - \dim\big(u(V)^\perp \cap \mathrm{Ann}(\iota^*(TM))\big) \\
&\qquad = \dim u(V)^\perp + \dim \mathrm{Ann}(\iota^*(TM)) - \big(\dim \mathrm{Ann}(\iota^*(TM)) - \dim u(v) + 1\big) \\
&\qquad = \dim u(V)^\perp + \dim u(v) - 1 = n - 1,
\end{aligned}$$

which establishes (6-4), so we are done. □

We can now easily reprove a partial converse to the cylinder deformation theorem, originally proved by Mirzakhani and Wright [16, Theorem 1.5]:

Theorem 6.2 *Let M be an affine invariant manifold in any holomorphic stratum. Let $(X,\omega) \in M$ and let $\mathcal{C}$ be a full equivalence class of horizontal M–parallel cylinders. Then, up to purely relative deformations, the only small deformations of (X,ω) that stay in M and are supported on the union of the cylinders in $\mathcal{C}$ are given by $a_t^{\mathcal{C}} u_s^{\mathcal{C}}(X,\omega)$ for small $t, s \in \mathbb{R}$.*

Proof By Lemma 6.1, the space of such deformations is at most one-dimensional. By the cylinder deformation theorem (Theorem 1.9), the deformation given by applying $a_t^{\mathcal{C}} u_s^{\mathcal{C}}$ lies in M. Hence, this one-complex dimensional family comprises all deformations of the specified type. □

For further use, we record the following easy statement:

Lemma 6.3 *The set of all horizontal vanishing cycles is linearly independent in punctured homology $H_1(X \setminus \underline{z};\mathbb{C})$.*

Proof For each horizontal vanishing cycle λ_e, we can choose a homology class $\delta \in H_1(X, \underline{z}; \mathbb{C})$ that intersects λ_e and no other horizontal vanishing cycle. For instance, δ can be constructed by locating two marked zeros at levels below $\ell(e)$ that can be connected by a path that crosses no horizontal nodes except e, and is contained in $X_{(\leq \ell(e))}$. The existence of such marked zeros is guaranteed by [2, Lemma 3.9]. □

We note that it is not true that the set of all vanishing cycles altogether is linearly independent in punctured homology. Indeed, if some irreducible component of the multiscale differential does not contain any marked zero, then the sum of the vanishing cycles that is its boundary is homologous to zero.

6.2 Minimal stratum

We now specialize to the case of affine invariant manifolds in a *minimal* holomorphic stratum $\Omega\mathcal{M}_{g,1}(2g-2)$, i.e. to the case when the differential has only zero, of maximal multiplicity. The special feature of the minimal stratum is that both maps ι and u above are isomorphisms; in particular, all horizontal vanishing cycles are linearly independent in the absolute homology $H_1(X; \mathbb{C})$. As always, we study the situation near some $p_0 \in \partial M \subset \Xi\overline{\mathcal{M}}_{g,1}(2g-2)$, and the first result we obtain is the following.

Proposition 6.4 *Let $e_1 \neq e_2 \in E^{\mathrm{hor}}(\Gamma)$ be M–cross-related horizontal nodes. Then there is a defining equation F of M that crosses e_1 and e_2 and no other horizontal nodes, i.e. $E^{\mathrm{hor}}(F) = \{e_1, e_2\}$.*

Proof Let λ_1 and λ_2 be the vanishing cycles for e_1 and e_2, and let Λ be the M–cross-equivalence class containing them. Let W be the span of the elements of Λ. By Lemma 6.3, $\dim W = |\Lambda|$. By Theorem 1.1, the vanishing cycles in Λ all have proportional periods on M, and so the corresponding cylinders are M–parallel. Now let V be the span of the vanishing cycles of all cylinders that are M–parallel to these. Since $W \subset V$, by Lemma 6.1 (and using that the map ι is an isomorphism, since we are working in the minimal stratum), we get

$$\dim(TM \cap \operatorname{Ann} W^{\perp}) \leq \dim(TM \cap \operatorname{Ann} V^{\perp}) \leq 1.$$

Hence,

$$\dim \operatorname{Ann} W^{\perp} - \dim(TM \cap \operatorname{Ann} W^{\perp}) \geq \dim \operatorname{Ann} W^{\perp} - 1 = |\Lambda| - 1.$$

The left-hand side above is equal to the number of equations in the rref basis that cross some vanishing cycle in Λ. Since no equation can cross exactly one element of Λ, we get the desired conclusion. □

Proposition 6.5 *Suppose $F = a_1\lambda_1 + \cdots + a_n\lambda_k$ is a defining equation of M at p, where the λ_i are some distinct horizontal vanishing cycles. Then F is a sum of defining equations of M that have the form of pairwise proportionalities $b_j\lambda_j = c_l\lambda_l$ for $1 \le j, l \le k$ and some $b_j, c_l \in \mathbb{R}$.*

Proof Let $\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \dots, \lambda_n$ be all the horizontal vanishing cycles. By Lemma 6.6, we can write

$$F = F_1 + \cdots + F_\ell,$$

where each F_i is a defining equation of M of the form $F = b\lambda_j + c\lambda_l$. Order the F_i in such a way that $F_1, \dots, F_{\ell'}$ are of the form $b\lambda_j + c\lambda_l$ with $1 \le j, l \le k$, and the remaining equations $F_{\ell'+1}, \dots, F_\ell$ are not of this form. If $\ell = \ell'$, then we are done, so suppose $\ell > \ell'$ and recall that $\lambda_1, \dots, \lambda_n$ are linearly independent by Lemma 6.3. If all the $F_{\ell'+1}, \dots, F_\ell$ are of the form $b\lambda_j + c\lambda_j$ with both $j, l > k$, then, since F itself does not have any λ_i terms with $i > k$, we must also have $F = F_1 + \cdots + F_{\ell'}$, and we are done. Otherwise, we can assume that some equation — without loss of generality $F_{\ell'+1}$ — is of the form $b\lambda_j + c\lambda_l$ with $j \le k < l$. In this case, since in the sum $F = F_1 + \cdots + F_\ell$ the λ_l terms must cancel out, some other equation — without loss of generality $F_{\ell'+2}$ — must have the form $b'\lambda_{j'} + c'\lambda_l$. We can then write the following new decomposition of F into defining equations:

$$\begin{aligned} F &= F_1 + \cdots + F_{\ell'} + \Big(F_{\ell'+1} + \frac{c'}{c}F_{\ell'+1}\Big) + \Big(F_{\ell'+2} - \frac{c'}{c}F_{\ell'+1}\Big) + F_{\ell'+3} + \cdots + F_\ell \\ &= F_1 + \cdots + F_{\ell'} + \Big(1 + \frac{c'}{c}\Big)F_{\ell'+1} + \Big(b'\lambda_{j'} - \frac{c'b}{c}\lambda_j\Big) + F_{\ell'+3} + \cdots + F_\ell. \end{aligned}$$

Note that the $(\ell'+2)^{\text{nd}}$ term now involves λ_j instead of λ_l. Thus, the total number of appearances of terms λ_l with $l > k$ has decreased by 1. Continuing in this fashion, we arrive at a decomposition with no such terms, and we are done. □

The proof above used the following statement:

Lemma 6.6 *Let $\lambda_1, \dots, \lambda_n$ be all the horizontal vanishing cycles. Then any defining equation of M of the form $F = a_1\lambda_1 + \cdots + a_n\lambda_n = 0$ is a sum of defining equations of M of the form $b\lambda_j + c\lambda_l$.*

Proof Consider an rref basis of defining equations of M. Suppose that, among these equations, $F_1, \dots, F_k$ are the ones that cross some vanishing cycle among $\lambda_1, \dots, \lambda_n$. We first claim that each such F_i must cross at least two of the these horizontal vanishing cycles. In fact, by Proposition 3.1, there is a linear relation among the vanishing cycles crossed by F_i. This could include vertical vanishing cycles, but, by Proposition 3.11(1), these all lie at lower level than the horizontal nodes crossed. Considering the limit of this relation as we approach p_0, and noting that the period of a horizontal vanishing cycle must be nonzero near p_0, we see that the relation must involve at least two horizontal vanishing cycles, as claimed.

Now, for each equation F_i, consider the pivot horizontal vanishing cycle λ_j for that equation, and choose some other horizontal vanishing cycle λ_l crossed by F_i (whose existence was just established). By Theorem 1.1, there is some equation $\alpha_i = b\lambda_j + c\lambda_l$ that holds on M. The $\alpha_1, \dots, \alpha_k$ must be linearly independent, since each involves a pivot node that doesn't appear in the others. Thus, we get k linearly independent relations, each establishing that periods over a pair of horizontal vanishing cycles are proportional.

Now let $\Lambda = \operatorname{span}(\lambda_1, \dots, \lambda_n)$. By [1], the tangent space $TM \subseteq H^1(X;\mathbb{C})$ is symplectic (here we again use that we are in the minimal stratum, so absolute and relative homology are the same). Hence, $\operatorname{Ann} TM$ is also symplectic, which implies that

$$\dim \Lambda \cap \operatorname{Ann} TM = \dim \operatorname{Ann} TM - \dim \Lambda^{\perp} \cap \operatorname{Ann} TM.$$

We see that the right-hand side equals k, since by assumption there are exactly k rref equations cutting out M that have nonzero intersection with an element of Λ. All $\alpha_1, \dots, \alpha_k$ lie in $\Lambda \cap \operatorname{Ann} TM$, and since they are linearly independent we have

$$\Lambda \cap \operatorname{Ann} TM = \operatorname{span}(\alpha_1, \dots, \alpha_k),$$

and we are done, since F also belongs in the left-hand space. □

Proof of Theorem 1.10 Part (1) will follow easily from Proposition 6.4. We argue by induction on n, the number of horizontal nodes crossed by the defining equation F. If $n \le 2$, there is nothing to prove. So suppose $n \ge 3$. First, consider the case where there exists a defining equation F' with $E^{\mathrm{hor}}(F') \subsetneq E^{\mathrm{hor}}(F)$ and $E^{\mathrm{hor}}(F') \neq \varnothing$. Subtracting from F a suitable multiple cF' gives an equation F'' that crosses a strictly smaller set of horizontal nodes than F. Applying the inductive hypothesis to F' and F'' gives that each can be written as a sum of defining equations that cross at most two horizontal nodes, and hence so can $F = cF' + F''$. On the other hand, in the case that no such F' exists, then any pair of horizontal nodes crossed by F are M–cross-related, by definition. Pick two of these nodes and call them e_1 and e_2. By Proposition 6.4, there is a defining equation F_0 with $E^{\mathrm{hor}}(F_0) = \{e_1, e_2\}$. Subtracting from F a suitable multiple cF_0 gives an equation F'' that crosses a strictly smaller set of horizontal nodes than F. By induction, this F'' can be written as a sum of defining equations that cross at most two horizontal nodes, and hence so can $F = cF_0 + F''$. This proves part (1).

Part (2) is just a restatement of Proposition 6.5. □

6.3 Counterexamples in general holomorphic strata

In this section, we show that analogs of certain results proved in Section 6.2 fail in general holomorphic strata with multiple zeros.

Example 6.7 We give an example that shows that Proposition 6.5 fails in general holomorphic strata. Define an affine invariant manifold $M \subset \Omega\mathcal{M}_{5,8}(1^8)$ as follows. Pick a surface $X \in \Omega\mathcal{M}_{2,2}(1,1)$, make two slits, take two copies of the slitted surface, and glue the surfaces together along both corresponding slits. This gives an X' that is a translation cover of X. Then M is defined as the connected component of X' in the space of translation covers of X with four branch points, none of which is a zero of the differential. See Figure 6 for a topological picture. Then M admits a degeneration p_0 where six classes in punctured homology, λ_1, $\lambda_{1'}$, λ_2, $\lambda_{2'}$, λ_3 and $\lambda_{3'}$, are vanishing cycles of horizontal nodes, as shown in Figure 6. Near this boundary point, M is cut out by the eight period equations shown next to the figure.

Now note that, taking the sum of the first three equations, and using that the relation $0 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_{1'} + \lambda_{2'} + \lambda_{3'}$ holds in absolute homology, we see that

$$\int_{2\lambda_1 + 2\lambda_2 + 2\lambda_3} \omega = 0$$

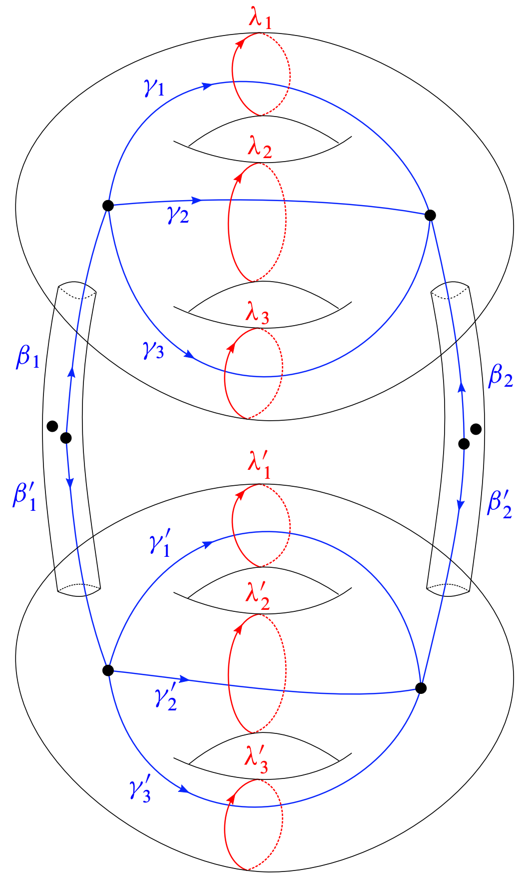
is an equation satisfied on M. However, no pair from λ_1, λ_2 and λ_3 need to have proportional periods on M, so the above equation cannot be written as a sum of pairwise equations among these three vanishing cycles. Hence, Proposition 6.5 fails for this M. ◁

Example 6.8 We now give a *local* counterexample M to Proposition 6.4 in a non-minimal stratum. The meaning of local here is that there exists $p_0 \in \partial \Xi \overline{\mathcal{M}}_{g,n}(\mu)$ and a neighborhood $U \ni p_0$ with a subvariety $M \subset U \cap \Omega\mathcal{M}_{g,n}(\mu)$ such that:

(1) At each point, M is locally cut out by linear equations in period coordinates.

(2) The closure $\overline{M}$ in U is an analytic subvariety of U, containing p_0.

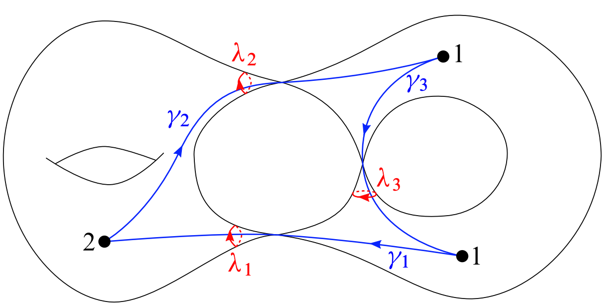
(3) $\iota^*(TM)$ is a symplectic subspace.



period equations cutting out M
$\int_{\lambda_1}\omega - \int_{\lambda_1'}\omega = 0$
$\int_{\lambda_2}\omega - \int_{\lambda_2'}\omega = 0$
$\int_{\lambda_3}\omega - \int_{\lambda_3'}\omega = 0$
$\int_{\gamma_1}\omega - \int_{\gamma_1'}\omega = 0$
$\int_{\gamma_2}\omega - \int_{\gamma_2'}\omega = 0$
$\int_{\gamma_3}\omega - \int_{\gamma_3'}\omega = 0$
$\int_{\beta_1}\omega - \int_{\beta_1'}\omega = 0$
$\int_{\beta_2}\omega - \int_{\beta_2'}\omega = 0$

Figure 6: Description of an affine invariant manifold M in the stratum $\Omega\mathcal{M}_{5,8}(1^8)$.

In other words, our M will be consistent with all the local analyticity and symplecticity properties used in the proofs above, but we do not claim that M is actually the intersection of some (global) affine invariant manifold with U.



period equations cutting out M
$\int_{\gamma_1}\omega + \int_{\gamma_2}\omega + \int_{\gamma_3}\omega = 0$
$\int_{\lambda_1}\omega + \int_{\lambda_2}\omega + \int_{\lambda_3}\omega = 0$

Figure 7: A local example in $\Omega\mathcal{M}_{3,3}(1,1,2)$.

We define $M \subset \Omega\mathcal{M}_{3,3}(1,1,2)$ according to Figure 7. Here the boundary point has three horizontal nodes, with vanishing cycles λ_1, λ_2 and λ_3, and no other nodes. (This does not specify a unique boundary point, since there is additional data about the geometry of the stable Riemann surface with differential. Any choice of this data will do.)

Note that (1) follows from the way we've defined M.

To verify condition (2), it suffices to convert the period coordinate equations into equations in analytic coordinates around p_0, using the ideas of Section 4. Indeed, the second equation, which involves the periods over horizontal vanishing cycles λ_i already extends to an analytic equation. The first equation, which involves the γ_i, can be written in terms of log periods as

$$\left(\Psi_{\gamma_1} + \int_{\lambda_1} \omega \cdot \ln s_1\right) + \left(\Psi_{\gamma_2} + \int_{\lambda_2} \omega \cdot \ln s_2\right) + \left(\Psi_{\gamma_3} + \int_{\lambda_3} \omega \cdot \ln s_3\right) = 0.$$

Since λ_1 and λ_2 are equal in absolutely homology, and using the second equation relating periods over λ_i, the above equation becomes

$$(\Psi_{\gamma_1} + \Psi_{\gamma_2} + \Psi_{\gamma_3}) + \left(\int_{\lambda_1} \omega\right)(\ln s_1 + \ln s_2 - 2\ln s_3) = 0.$$

Exponentiating both sides and rearranging gives

$$\exp(\Psi_{\gamma_1} + \Psi_{\gamma_2} + \Psi_{\gamma_3}) s_1 s_2 - s_3^2 = 0.$$

This is an analytic equation in analytic coordinates around p_0, satisfied at the point p_0 where $s_1 = s_2 = s_3 = 0$ at p_0. This establishes (2).

For (3), observe that the two equations, corresponding to $\gamma_1 + \gamma_2 + \gamma_3$ and λ_1, λ_2 and λ_3, are both absolute homology classes, and their intersection pairing is nonzero.

Having verified that M is in fact a local example, we now show that Proposition 6.4 does not hold in this setting. Indeed, the first defining equation of M crosses all three horizontal vanishing cycles λ_1, λ_2 and λ_3, and is the only defining equation of M that crosses any of these three horizontal vanishing cycles. Hence, λ_1 and λ_2 are M–cross-related, but there is no defining equation F that crosses λ_1 and λ_2 but not λ_3. ◁

6.4 Boundary stratification of affine invariant manifolds

For potential use for classifying affine invariant manifolds by recursively applying degeneration techniques, we record here two results on their boundaries. The first result applies more generally to real-linear subvarieties of meromorphic strata — the context in which our generalization of the cylinder deformation theorem applies.

Proposition 6.9 *Let M be a linear subvariety defined by equations with all coefficients real. Then, for any Γ such that $\dim \partial M_\Gamma > 0$, the boundary stratum $\partial M_\Gamma \subset \overline{M}$ is not compact.*

Proof By [4], the boundary ∂M_Γ is a product of linear subvarieties of various strata of differentials, also all defined by linear equations with real coefficients. Suppose for contradiction that $\dim \partial M_\Gamma$ is compact. Then, for any irreducible component X_v of X, the image of ∂M_Γ under the projection to the moduli of differentials on the corresponding component would have to be compact. Since $\dim \partial M_\Gamma > 0$, there must exist some X_v such that the image of ∂M_Γ gives a positive-dimensional linear subvariety of the space of differentials on that component. If the differential considered on X_v is holomorphic, we claim that this is impossible since there does not exist any compact $\mathrm{GL}^+(2,\mathbb{R})$ affine invariant manifold (because we can choose a saddle connection and make it shrink to zero under the action of $\mathrm{SL}(2,\mathbb{R})$). If the differential considered on X_v is meromorphic, then by [7] the corresponding stratum of differentials does not contain any compact complex subvarieties. Thus, in either case we have a contradiction. □

While we do not know a counterexample to this proposition for linear subvarieties defined by equations with complex coefficients, note that the proof uses the $\mathrm{GL}^+(2,\mathbb{R})$–action for holomorphic components, as it is not known whether holomorphic strata contain any compact complex subvarieties. Even if we start with a linear subvariety of a meromorphic stratum, it could be that in the boundary there is a top-level component where the differential is holomorphic.

Applying this proposition recursively, given a real-linear subvariety M of complex dimension a, one can consider a Γ_1 that corresponds to a divisorial boundary component, that is such that $\dim_{\mathbb{C}} \partial M_{\Gamma_1} = a-1$, and then, since ∂M_Γ is not compact, consider Γ_2 such that $\dim_{\mathbb{C}} \partial M_{\Gamma_2} = a-2$, and so on, thus constructing a sequence of divisorial degenerations of length precisely a:

$$\mathrm{pt} = \Gamma_0 \rightsquigarrow \cdots \rightsquigarrow \Gamma_a \quad \text{such that} \quad \dim_{\mathbb{C}} \partial M_{\Gamma_i} = a-i.$$

Compare this to Remark 3.4, where many sequences of divisorial degenerations are constructed, *starting from a given* Γ. Here we claim that, choosing at each step any divisorial boundary component, we can always construct a sequence of degenerations of length a, i.e. going down to a point. For the case of affine invariant manifolds, we can say a bit more:

Corollary 6.10 *Let M be an affine invariant manifold, i.e. a real-linear subvariety of a holomorphic stratum. If ∂M_Γ is a deepest stratum of $\overline{M}$, that is, $\partial M_{\Gamma'} = \varnothing$ for any degeneration $\Gamma \rightsquigarrow \Gamma'$, then every top-level vertex of Γ has a horizontal edge attached to it.*

Proof By the proposition, being deepest is equivalent to ∂M_Γ simply being a point. As in the proof of the proposition, the projection of ∂M_Γ to the stratum corresponding to some top-level component X_v must be a real-linear subvariety of that stratum, which in this case must be just one point. If X_v has no horizontal nodes, then the twisted differential η_v on X_v has no poles: it has zeroes at any marked points z_i, and possibly zeroes at the vertical nodes. But then, again as in the proof above, there does not exist any flat surface in a holomorphic stratum fixed by $\mathrm{GL}^+(2,\mathbb{R})$, so we have a contradiction. □

We note that this corollary is false for real-linear subvarieties of meromorphic strata. For example, for the case of the closure of the Hurwitz space considered in Section 4.3, the boundary points have no horizontal nodes whatsoever. This is also the case for the closure of the double ramification cycle considered in [5].

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