Solutions to Exam 1 Practice Problems

Math 352, Fall 2014

1. (a) $s'(t) = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(1-\cos t)^2 + (-\sin t)^2} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$

$= \boxed{\sqrt{2 - 2\cos t}}$.

(b) $$\kappa_g(t) = \frac{-x''(t)y'(t) + y''(t)x'(t)}{s'(t)^3} = \frac{-(\sin t)(-\sin t) + (-\cos t)(1-\cos t)}{(2-2\cos t)^{3/2}}$$

$$= \frac{1-\cos t}{2^{3/2}(1-\cos t)^{3/2}} = \boxed{\frac{1}{2\sqrt{2-2\cos t}}}$$

(c) $$\int_{\mathcal{C}} \sqrt{1-y^2}\, ds = \int_0^{\pi/2} \sqrt{1-y(t)^2}\, s'(t)\, dt = \int_0^{\pi/2} \sin t\, \sqrt{2-2\cos t}\, dt$$

$$= \left[\frac{1}{3}(2-2\cos t)^{3/2}\right]_0^{\pi/2} = \boxed{\frac{2\sqrt{2}}{3}}$$

(d) $$\int_{\mathcal{C}} y\, dx = \int_0^{\pi/2} y(t)\, x'(t)\, dt = \int_0^{\pi/2} (\cos t)(1-\cos t)\, dt = \int_0^{\pi/2} (\cos t - \cos^2 t)\, dt$$

$$= \int_0^{\pi/2} \left(\cos t - \frac{1+\cos(2t)}{2}\right) = \left[\sin t - \frac{t}{2} - \frac{\sin(2t)}{4}\right]_0^{\pi/2} = \boxed{1 - \frac{\pi}{4}}$$

2. The curve $\mathcal{C}$ surrounding the shaded region corresponds to $-\pi/2 \le t \le \pi/2$. Thus

$$\text{area} = \int_{\mathcal{C}} -y\, dx = \int_{-\pi/2}^{\pi/2} -y(t)\, x'(t)\, dt = \int_{-\pi/2}^{\pi/2} -(\sin^3 t \cos t)(-\sin t)\, dt$$

$$= \int_{-\pi/2}^{\pi/2} \sin^4 t \cos t\, dt = \left[\frac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = \boxed{\frac{2}{5}}$$

3. Let $\mathcal{R}$ be the region inside the curve. By Green's Theorem,

$$\int_{\mathcal{C}} \big(\sin(x^2) + 2y\big)\, dx + 5x\, dy = \iint_{\mathcal{R}} \left(\frac{\partial}{\partial x}[5x] - \frac{\partial}{\partial y}[\sin(x^2) + 2y]\right) dA$$

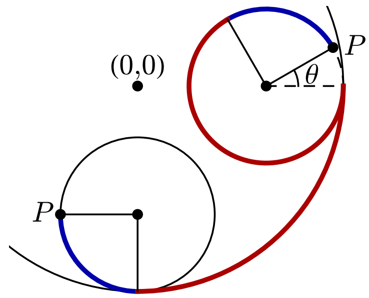
$$= \iint_{\mathcal{R}} 3\, dA = 3\,\text{area}(\mathcal{R}) = 3(21) = \boxed{63}$$

4. Note that this curve begins at $(1,0)$ and ends at $(0,2)$. Note also that

$$e^y = \frac{\partial}{\partial x}\left[xe^y + \frac{y^3}{3}\right] \qquad \text{and} \qquad xe^y + y^2 = \frac{\partial}{\partial y}\left[xe^y + \frac{y^3}{3}\right].$$

Then $$\int_{\mathcal{C}} e^y\, dx + \big(xe^y + y^2\big)\, dy = \left[xe^y + \frac{y^3}{3}\right]_{(1,0)}^{(0,2)} = \frac{8}{3} - 1 = \boxed{\frac{5}{3}}$$

5. The distance rolled on the large circle must equal the distance rolled on the small circle:



The large circle has a circumferance of 32π, so the red arc has a length of 8π. The length of the red arc on the small circle must be the same. The small circle has a circumferance of 12π, so the red arc is 2/3 of this circle, or $4\pi/3$ radians. The blue arc subtends an angle of $\pi/2$ radians, and therefore $\theta = 2\pi - (4\pi/3) - (\pi/2) = \pi/6$. Then

$$P = (10,0) + 6(\cos\pi/6, \sin\pi/6) = \boxed{\left(10+3\sqrt{3},\, 1/2\right)}$$

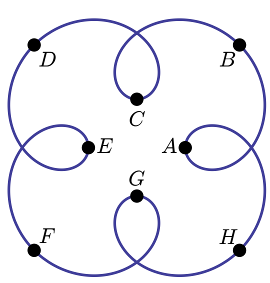
6. (a) We have

$$\begin{aligned} s(t) &= \int \sqrt{x'(t)^2 + y'(t)^2}\,dt \\ &= \int \sqrt{(-t^2\sin t + 2t\cos t)^2 + (t^2\cos t + 2t\sin t)^2}\,dt \\ &= \int \sqrt{t^4\sin^2 t - 2t^3\sin t\cos t + 4t^2\cos^2 t + t^4\cos^2 t + 2t^3\sin t\cos t + 4t^2\sin^2 t}\,dt \\ &= \int \sqrt{t^4 + 4t^2}\,dt = \int t\sqrt{t^2+4}\,dt = \boxed{\frac{\left(t^2+4\right)^{3/2}}{3}} \end{aligned}$$

(b) Solving the equation $s = \left(t^2+4\right)^{3/2}/3$ for t gives $t = \sqrt{(3s)^{2/3} - 4}$. Note that $s = 8/3$ when $t = 0$. Therefore, the desired parametrization is

$$\boxed{\left(\left((3s)^{2/3}-4\right)\cos\sqrt{(3s)^{2/3}-4},\ \left((3s)^{2/3}-4\right)\sin\sqrt{(3s)^{2/3}-4}\right) \quad \text{for } s > \frac{8}{3}}$$

7. (a) The curve clearly has eight vertices:



The coordinates for the first two are $\boxed{A = \vec{x}(0) = (1,0)}$ and $\boxed{B = \vec{x}(\pi/4) = (3/\sqrt{2}, 3/\sqrt{2})}$, and the remaining six are symmetric with these.

(b) Observe that

$$\begin{aligned} \vec{x}\,'(t) &= (-2\sin t + 5\sin 5t, 2\cos t - 5\cos 5t) \\ \text{and} \qquad \vec{x}\,''(t) &= (-2\cos t + 25\cos 5t, -2\sin t + 25\sin 5t). \end{aligned}$$

Vertex A is at $t = 0$, with

$$\vec{x}\,'(0) = (0, -3) \qquad \text{and} \qquad \vec{x}\,''(0) = (23, 0).$$

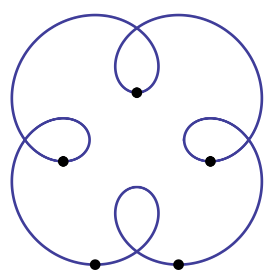
Thus $\kappa_g(0) = \dfrac{\vec{x}\,'' \cdot \vec{U}}{(s')^2} = \boxed{\dfrac{23}{9}}$ at vertex A. The curvature is the same at vertices C, E, and G.

Vertex B is at $t = \pi/4$, with

$$\vec{x}\,'(\pi/4) = \frac{1}{\sqrt{2}}(-7, 7) \qquad \text{and} \qquad \vec{x}\,''(\pi/4) = \frac{1}{\sqrt{2}}(-27, -27).$$

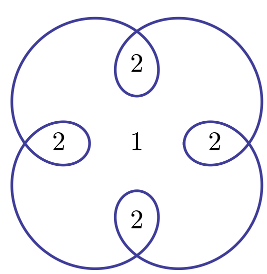
Thus $\kappa_g(0) = \dfrac{\vec{x}\,'' \cdot \vec{U}}{(s')^2} = \boxed{\dfrac{27}{49}}$ at vertex B. The curvature is the same at vertices D, F, and H.

(c) Since $\theta(t)$ is increasing, we can count the rotation index by finding all the points for which $\theta = 0$.



The curve makes one full rotation between every pair of these points. As you can see, the rotation index is 5, and therefore $\int_{\mathcal{C}} \kappa_g \, ds = \boxed{10\pi}$.

(d) The following picture shows the winding number of the curve around each of the five bounded regions.



It follows that

$$\int_{\mathcal{C}} x\,dy \;=\; 21.0 + 2(0.9) + 2(0.9) + 2(0.9) + 2(0.9) \;=\; \boxed{28.2}$$

8. We have

$$\theta(s) \;=\; \int \kappa_g(s)\,ds \;=\; \int \frac{ds}{\sqrt{1-s^2}} \;=\; \sin^{-1}(s) + C.$$

Since $\vec{x}'(0) = (1,0)$, we know that $\theta(0) = 0$, so $\theta(s) = \sin^{-1}(s)$. Then

$$\vec{x}(s) \;=\; \int \big(\cos\theta(s), \sin\theta(s)\big)\,ds \;=\; \int \big(\sqrt{1-s^2}, s\big)\,ds.$$

The integral of $\sqrt{1-s^2}$ is difficult. Substituting $s = \sin\phi$ gives

$$\int \sqrt{1-s^2}\,ds \;=\; \int \cos^2\phi\,d\phi \;=\; \int \frac{1+\cos(2\phi)}{2}\,d\phi \;=\; \frac{\phi}{2} + \frac{\sin(2\phi)}{4} + C$$
$$\;=\; \frac{\phi}{2} + \frac{\sin\phi\cos\phi}{2} + C \;=\; \frac{\sin^{-1}(s)}{2} + \frac{s\sqrt{1-s^2}}{2} + C.$$

So $\vec{x}(s) \;=\; \left(\dfrac{\sin^{-1}(s) + s\sqrt{1-s^2}}{2}, \dfrac{s^2}{2}\right) + \vec{C}$. Since $\vec{x}(0) = (0,0)$, it follows that $\vec{C} = (0,0)$ so $\boxed{\vec{x}(s) \;=\; \left(\dfrac{\sin^{-1}(s) + s\sqrt{1-s^2}}{2}, \dfrac{s^2}{2}\right)}$

9. The equation for $L(t)$ is $y = (-\cot t)x + \cos t$, so $L(t)$ and $L(t+h)$ intersect when

$$\big(-\cot(t+h)\big)x + \cos(t+h) \;=\; (-\cot t)x + \cos t.$$

Solving for x gives

$$x \;=\; \frac{\cos(t+h) - \cos t}{\cot(t+h) - \cot t}.$$

We can now use L'Hôpital's rule to take the limit as $h \to 0$:

$$x \;=\; \lim_{h\to 0} \frac{\cos(t+h) - \cos t}{\cot(t+h) - \cot t} \;=\; \lim_{h\to 0} \frac{-\sin(t+h)}{-\csc^2(t+h)} \;=\; \sin^3 t.$$

Pugging this into the formula for $L(t)$ gives $y = \cos^3 t$. Thus $\boxed{\vec{x}(t) = \big(\sin^3 t, \cos^3 t\big)}$

10. (a) The tangent vector is $\vec{T} = (4/5, 3/5)$, so the normal vector is $\vec{U} = (-3/5, 4/5)$. The osculating circle has a radius of 2, so it is centered at $(2,2) + 2\vec{U} = (4/5, 18/5)$. Thus, the equation is $\boxed{(x-4/5)^2 + (y-18/5)^2 = 4}$

(b) We know that $\theta'(t) = \dfrac{d\theta}{dt} = \dfrac{d\theta}{ds}\dfrac{ds}{dt} = \kappa_g(t)\, s'(t)$. Then $\theta'(0) = \kappa_g(0)\, s'(0) = (1/2)(5) = \boxed{5/2}$

(c) $\vec{x}\,'' = (s')^2\,\kappa_g\,\vec{U} + s''\,\vec{V} = (5)^2(1/2)(-3/5, 4/5) + (1)(-3/5, 4/5) = \boxed{(-8.1, 10.8)}$

11. The vector $\vec{w} = \dfrac{1}{\sqrt{5}}(2, -1)$ points along the axis of the second parabola, and the vector $\vec{v} = \dfrac{1}{\sqrt{5}}(-1, -2)$ is 90° clockwise from this. Then the second parabola is given by

$$\vec{x}(t) \;=\; t\,\vec{v} + t^2\,\vec{w} \;=\; \boxed{\frac{1}{\sqrt{5}}\big(2t^2 - t,\; -t^2 - 2t\big)}$$

12. Since $\kappa_g(t) = 1$, this curve must be a circle (or arc of a circle) of radius 1. Since $\vec{x}(0) = (1, 0)$ and $\vec{x}\,'(0) = (0, 1)$, it is in fact the unit circle centered at the origin. Thus, the parameterization looks like

$$\vec{x}(t) \;=\; \big(\cos f(t), \sin f(t)\big)$$

for some function $f(t)$. It's easy to see (either intuitively or by doing a calculation), that $s'(t) = f'(t)$, so

$$f(t) \;=\; \int s'(t)\,dt \;=\; \int (1 + t^2)\,dt \;=\; t + \frac{t^3}{3} + C.$$

Since $\vec{x}(0) = (1, 0)$, we can assume that $f(0) = 0$, and hence $C = 0$. Thus

$$\boxed{\vec{x}(t) \;=\; \left(\cos\left(t + \frac{t^3}{3}\right), \sin\left(t + \frac{t^3}{3}\right)\right)}$$