Exam 1 Practice Problems

Math 352, Fall 2014

1. Let $\mathcal{C}$ be the cycloid $\vec{x}(t) = (t - \sin t, \cos t)$ for $0 \leq t \leq \pi/2$.

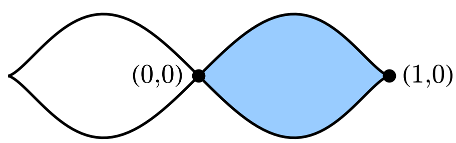
   (a) Compute the speed function $s'(t)$ for this parametrization.

   (b) Compute the curvature $\kappa_g(t)$.

   (c) Evaluate $\displaystyle\int_{\mathcal{C}} \sqrt{1-y^2}\, ds$.

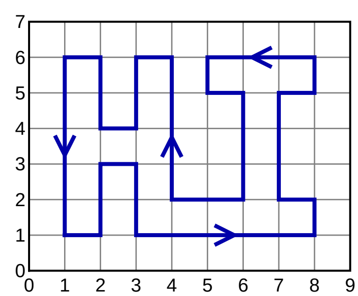
   (d) Evaluate $\displaystyle\int_{\mathcal{C}} y\, dx$.

2. The following figure shows the curve $\vec{x}(t) = \big(\cos t, \sin^3 t \cos t\big)$.



   Find the area of the shaded region.

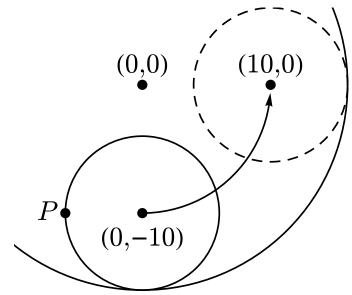
3. Evaluate $\displaystyle\int_{\mathcal{C}} \big(\sin(x^2) + 2y\big)\, dx + 5x\, dy$, where $\mathcal{C}$ is the closed curve shown in the following figure.



4. Evaluate $\displaystyle\int_{\mathcal{C}} e^y\, dx + \big(xe^y + y^2\big)\, dy$, where $\mathcal{C}$ is the curve

$$\vec{x}(t) = \Big((1-t)e^{\sqrt{t}},\ 2\sin(\pi t^2/2)\Big), \qquad 0 \leq t \leq 1.$$

5. A circle of radius 6 is rolling inside a large circle of radius 16.



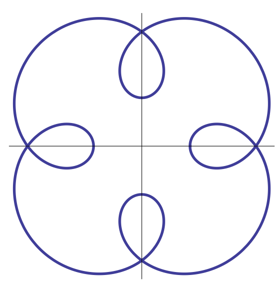
The large circle is centered at $(0,0)$, and the rolling circle is initially centered $(0,-10)$. Let P be the point on the rolling circle with initial coordinates $(-6,-10)$. Find the coordinates of P when the center of the rolling circle reaches the point $(10,0)$.

6. Let $\mathcal{C}$ be the curve $\vec{x}(t) = (t^2 \cos t, t^2 \sin t)$ for $t > 0$.

   (a) Find an arc length function $s(t)$ for this curve.

   (b) Find a unit-speed parametrization for this curve. Make sure to indicate the allowed values for the new parameter.

7. Let $\mathcal{C}$ be the curve $\vec{x}(t) = (2\cos t - \cos 5t,\, 2\sin t - \sin 5t)$, as shown in the following figure.



   (a) Based on the figure, how many vertices does $\mathcal{C}$ have? Find the coordinates for each vertex.

   (b) Determine the curvature of $\mathcal{C}$ at each of its vertices.

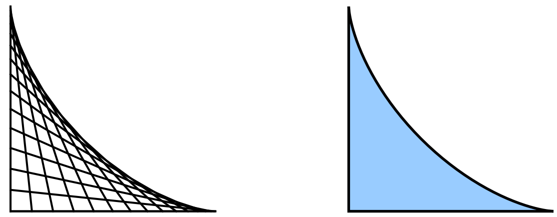
   (c) What is the value of $\displaystyle\int_{\mathcal{C}} \kappa_g(s)\, ds$?

   (d) The complement $\mathbb{R}^2 - \mathcal{C}$ has one unbounded component, one component with an area of approximately 21.0, and four components with areas of approximately 0.9 each. Use this information to estimate the value of $\displaystyle\int_{\mathcal{C}} x\, dy$.

8. Find a unit-speed parametric curve $\vec{x}(s)$ (for $-1 < s < 1$) such that

$$\vec{x}(0) = (0,0), \qquad \vec{x}\,'(0) = (1,0), \qquad \text{and} \qquad \kappa_g(s) = \frac{1}{\sqrt{1-s^2}}.$$

9. For $0 \le t \le \pi/2$, let $L(t)$ be the line segment from $(0, \cos t)$ to $(\sin t, 0)$, and let $\mathcal{R}$ be the union of all such lines, as shown in the figure below



Find parametric equations for the top curve of $\mathcal{R}$.

10. A regular parametric curve $\vec{x}(t)$ satisfies the conditions

$$\vec{x}(0) = (2, 2), \qquad \vec{x}\,'(0) = (4, 3), \qquad s''(0) = 1, \qquad \text{and} \qquad \kappa_g(0) = 1/2,$$

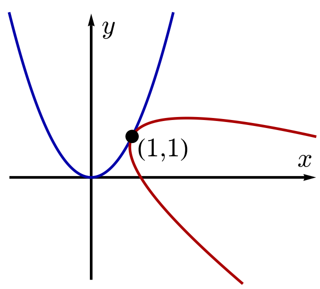
where $s(t)$ is an arc length parameter, and $\kappa_g(t)$ is the curvature at time t.

(a) Find the equation of the osculating circle to the curve at $t = 0$.

(b) Find $\theta'(0)$, where $\theta(t)$ denotes the direction of the unit tangent vector at time t.

(c) Find the value of $\vec{x}\,''(0)$.

11. The following picture shows two identical parabolas that are tangent at the point $(1, 1)$.



The first parabola is $y = x^2$, and the second parabola has a vertex at $(1, 1)$. Find parametric equations for the second parabola.

12. A regular parametric curve $\vec{x}(t)$ satisfies the initial conditions

$$\vec{x}(0) = (1, 0) \qquad \text{and} \qquad \vec{x}\,'(0) = (0, 1).$$

In addition,

$$s'(t) = 1 + t^2, \qquad \text{and} \qquad \kappa_g(t) = 1,$$

for all t, where $s(t)$ denotes an arc length parameter, and $\kappa_g(t)$ denotes the curvature. Use this information to find the formula for $\vec{x}(t)$.