Math 352 Name: SOLUTION

Exam 1

1. [**8 points**] Find a unit-speed parametrization for the curve $r = e^{2\theta}$.

$$\vec{x}(t) = (e^{2t}\cos t,\ e^{2t}\sin t)$$

$$\vec{x}'(t) = e^{2t}(-\sin t, \cos t) + 2e^{2t}(\cos t, \sin t)$$

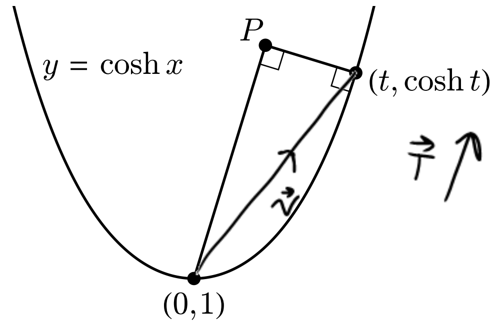
$$\|\vec{x}'(t)\| = \sqrt{(e^{2t})^2 + (2e^{2t})^2} = \sqrt{5}\, e^{2t}$$

$$s(t) = \int \sqrt{5}\, e^{2t}\, dt = \frac{\sqrt{5}}{2} e^{2t}$$

$$t = \frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}\right)$$

$$\boxed{\left(\frac{2s}{\sqrt{5}}\cos\left(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}\right)\right),\ \frac{2s}{\sqrt{5}}\sin\left(\frac{1}{2}\log\left(\frac{2s}{\sqrt{5}}\right)\right)\right)} \quad s>0$$

2. [**10 points**] In the following figure, find the coordinates of the point P in terms of t.



$\vec{T} \nearrow (\operatorname{sech} t, \tanh t)$

$$\vec{v} = (t, \cosh t - 1)$$

$$\vec{v} \cdot \vec{T} = t \operatorname{sech} t + (\cosh t - 1) \tanh t$$

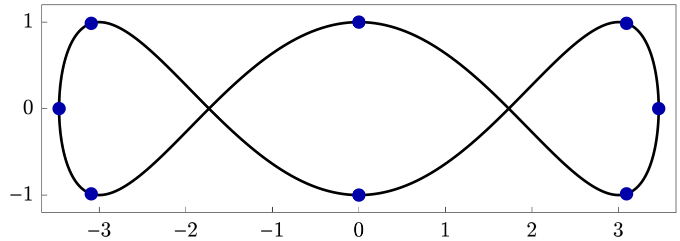
$$= t \operatorname{sech} t + \sinh t - \tanh t$$

$$P = (0,1) + (\vec{v} \cdot \vec{T}) \vec{T}$$

$$= (0,1) + (t \operatorname{sech} t + \sinh t - \tanh t)(\operatorname{sech} t, \tanh t)$$

$$= \boxed{\begin{array}{l}(t \operatorname{sech}^2 t + \tanh t - \operatorname{sech} t \tanh t, \\ \quad t \operatorname{sech} t \tanh t + \operatorname{sech}^2 t + \sinh t \tanh t)\end{array}}$$

3. **[36 points]** The following picture shows the curve $\vec{x}(t) = \left(2\sqrt{3}\cos t,\, \sin 3t\right)$.



(a) **[4 pts]** Compute the unit tangent vector to this curve at $t = \pi/3$.

$\vec{x}'(t) = (-2\sqrt{3}\sin t,\ 3\cos 3t)$

$\vec{x}'(\pi/3) = (-3, -3)$

$\vec{T}(\pi/3) = \boxed{\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)}$

(b) **[8 pts]** Find the Cartesian equation of the osculating circle to this curve at the point $(0, -1)$.

$\vec{x}'(\pi/2) = (-2\sqrt{3}, 0)$ $\quad s'(\pi/2) = 2\sqrt{3}$

$\vec{x}''(t) = (-2\sqrt{3}\cos t, -9\sin 3t)$

$\vec{x}''(\pi/2) = (0, 9)$ $\quad \vec{u}(\pi/2) = (0, -1)$

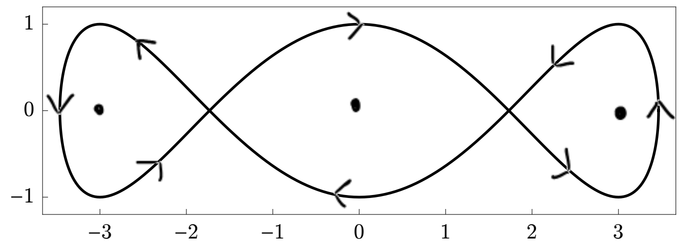
$\kappa_g = \dfrac{\vec{x}'' \cdot \vec{u}}{s'(t)^2} = \dfrac{-9}{(2\sqrt{3})^2} = -\dfrac{3}{4} \Rightarrow r = 4/3$

$\boxed{x^2 + (y - 1/3)^2 = 16/9}$

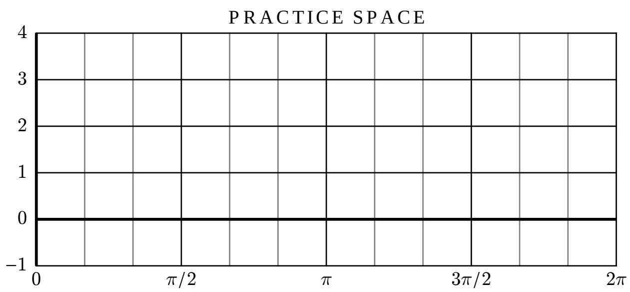
(c) **[4 pts]** Based on the given picture, how many vertices does the curve have? Draw points on the picture showing the approximate positions of these vertices.

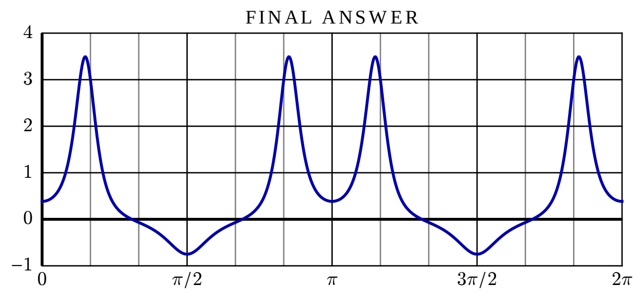
8 vertices (see above)

The following picture shows the curve $\vec{x}(t) = \left(2\sqrt{3}\cos t,\, \sin 3t\right)$.



(d) **[8 pts]** Use the picture to make a rough sketch of the curvature function $\kappa_g(t)$.





(e) **[4 pts]** Evaluate $\int_C \kappa_g(s)\, ds$.

$2\pi \times (\text{rotation index})$

$= 2\pi \times 1$

$= \boxed{2\pi}$

(f) **[4 pts]** Determine the winding number of this curve around each of the following points: $(3, 0)$, $(0, 0)$, and $(-3, 0)$.

$(3,0)$: 1

$(0,0)$: -1

$(-3,0)$: 1

(g) **[4 pts]** The value of $\int_C x\, dy$ is (choose one):

It's 0, actually.

(a) less than -5 (b) between -5 and -1 (c) between -1 and 1

(d) between 1 and 3 (e) between 3 and 5 (f) greater than 5

4. [**6 points**] Evaluate $\int_{\mathcal{C}} 3x^2\cos(y^2)\,dx \;-\; 2x^3 y\sin(y^2)\,dy$, where $\mathcal{C}$ is any curve from the point $(1, \sqrt{\pi})$ to the point $(2, 0)$.

$$3x^2\cos(y^2) = \frac{\partial}{\partial x}\left[x^3\cos(y^2)\right]$$

$$-2x^3 y\sin(y^2) = \frac{\partial}{\partial y}\left[x^3\cos(y^2)\right]$$

$$\left[x^3\cos(y^2)\right]_{(1,\sqrt{\pi})}^{(2,0)} = 8 - (-1) = \boxed{9}$$

5. [**8 points**] Suppose that a regular parametric curve $\vec{x}(t)$ has curvature $\kappa_g(t) = 3t^2$ and speed $s'(t) = 2t^3 + 1$. Given that $\vec{x}\,'(0) = (0, 1)$, find a formula for the velocity $\vec{x}\,'(t)$ as a function of t.

$$\theta(t) = \int \kappa_g\,ds = \int \kappa_g(t)\,s'(t)\,dt$$

$$= \int 3t^2(2t^3+1)\,dt$$

$$= \int (6t^5 + 3t^2)\,dt$$

$$= t^6 + t^3 + C$$

$\vec{x}\,'(0) = (0,1)$

so $\theta(0) = \pi/2$

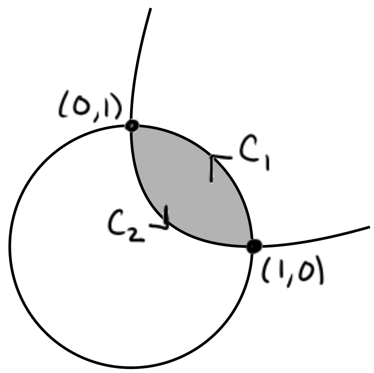
so $C = \pi/2$

$$\vec{x}\,'(t) = s'(t)\left(\cos\theta(t), \sin\theta(t)\right)$$

$$= (2t^3+1)\left(\cos\left(t^6+t^3+\tfrac{\pi}{2}\right), \sin\left(t^6+t^3+\tfrac{\pi}{2}\right)\right)$$

$$= \boxed{(2t^3+1)\left(-\sin(t^6+t^3), \cos(t^6+t^3)\right)}$$

6. [**12 points**] Use Green's Theorem to evaluate $\iint_{\mathcal{R}} x\,dA$, where $\mathcal{R}$ is the region bounded by the circle $x^2+y^2=1$ and the curve $\vec{x}(t)=\big(t^2,(1-t)^2\big)$ shown in the following figure.



$$\iint_R x\,dA = \int_C -xy\,dx$$

$$= \int_{C_1} -xy\,dx + \int_{C_2} -xy\,dx$$

$x=\cos t$ $\quad$ $dx = -\sin t\,dt$ $\quad$ $y=\sin t$

$x=t^2$ $\quad$ $dx = 2t\,dt$ $\quad$ $y=(1-t)^2$

$$= \int_0^{\pi/2} \sin^2 t\cos t\,dt - \int_0^1 2t^3(1-t)^2\,dt$$

$$= \left[\tfrac{1}{3}\sin^3 t\right]_0^{\pi/2} - \int_0^1 2t^3(1-2t+t^2)\,dt$$

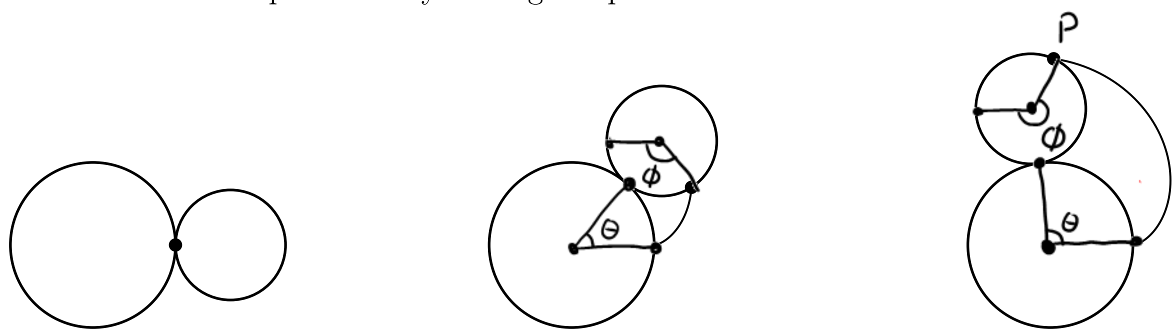
$$= \frac{1}{3} - \int_0^1 (2t^3-4t^4+2t^5)\,dt$$

$$= \frac{1}{3} - \left[\tfrac{1}{2}t^4 - \tfrac{4}{5}t^5 + \tfrac{1}{3}t^6\right]_0^1$$

$$= \frac{1}{3} - \left(\tfrac{1}{2} - \tfrac{4}{5} + \tfrac{1}{3}\right)$$

$$= \boxed{\frac{3}{10}}$$

7. **[14 points]** A circle of radius $2/3$ is rolling counterclockwise around the unit circle $x^2+y^2=1$. A point P lies on the perimeter of the rolling circle, with initial coordinates $(1,0)$. Find parametric equations for the curve produced by tracing the path of P.

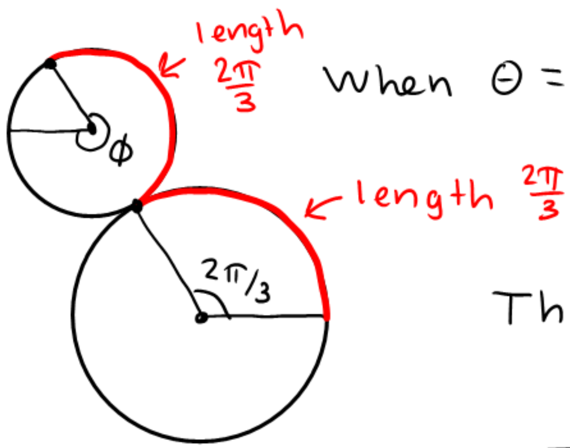


center of small circle $= \frac{5}{3}(\cos\theta, \sin\theta)$

The point P starts on the left of the small circle and rotates counterclockwise.

So $P = \frac{5}{3}(\cos\theta, \sin\theta) - \frac{2}{3}(\cos\phi, \sin\phi)$

But what is ϕ? It should be a multiple of θ.

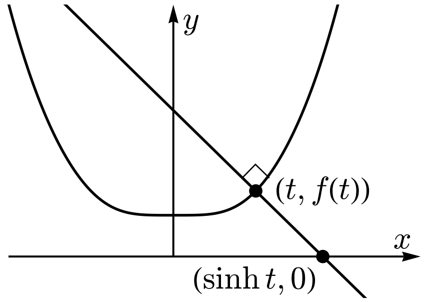


When $\theta = \frac{2\pi}{3}$, half the smaller circle has been used, so $\phi = \frac{5\pi}{3}$.

Thus, $\phi = \frac{5}{2}\theta$.

So $\boxed{P = \frac{5}{3}(\cos\theta, \sin\theta) - \frac{2}{3}\left(\cos\frac{5}{2}\theta, \sin\frac{5}{2}\theta\right)}$

8. [**6 points**] For a certain differentiable function $f(x)$, the normal line to the graph at each point $\big(t, f(t)\big)$ passes through the x-axis at the point $(\sinh t, 0)$.



Given that $f(0) = 1$, find a formula for f.

$$\text{Slope} = -\frac{1}{f'(t)} = \frac{f(t) - 0}{t - \sinh t}$$

Differential Equation:

$$-\frac{1}{y'} = \frac{y}{t - \sinh t}$$

$$y\,y' = \sinh t - t$$

$$\int y\,dy = \int (\sinh t - t)\,dt$$

$$\frac{1}{2}y^2 = \cosh t - \frac{1}{2}t^2 + C$$

$y(0) = 1$, so $C = -1/2$

$$y^2 = 2\cosh t - t^2 - 1$$

$$\boxed{f(t) = \sqrt{2\cosh t - t^2 - 1}}$$