Solutions to Exam 2 Practice Problems

Math 352, Fall 2014

Parameterizing Space Curves

1. After the rotation, the circle will still be centered at the origin, but will be in the *plane* $y = x$. The unit vectors $\vec{u}_1 = \frac{1}{\sqrt{2}}(1,1,0)$ and $\vec{u}_2 = (0,0,1)$ are orthogonal and parallel to this plane, so the circle is

$$\vec{x}(t) \;=\; (\cos t)\vec{u}_1 + (\sin t)\vec{u}_2 \;=\; \boxed{\frac{\cos t}{\sqrt{2}}(1,1,0) + \sin t\,(0,0,1)}$$

2. The point $\vec{P} = (t, \cosh t, 0)$ lies on the given catenary, and the unit normal vector to the given plane is $\vec{n} = \frac{1}{\sqrt{10}}(0,1,-3)$. Since the plane goes through the origin, all we have to do is project $\vec{P}$ onto $\vec{n}$:

$$\vec{v} \;=\; (\vec{P}\cdot\vec{n})\vec{n} \;=\; \frac{1}{10}\big((t,\cosh t,0)\cdot(0,1,-3)\big)(0,1,-3) \;=\; \frac{\cosh t}{10}(0,1,-3).$$

(Note how we handled the radical.) Then a point on the reflection is given by

$$\vec{P} + 2\vec{v} \;=\; \boxed{(t,\cosh t,0) + \frac{\cosh t}{5}(0,1,-3)}$$

3. This is just $\boxed{\vec{x}(t) = (\cos t)(2,4,4) + (\sin t)(2,1,-2)}$

4. The unit vectors $\vec{u}_1 = \frac{1}{\sqrt{5}}(1,2,0)$ and $\vec{u}_2 = (0,0,1)$ are orthogonal and parallel to this plane. Then the circle is

$$\vec{x}(t) \;=\; (1,2,1) + (\cos t)\vec{u}_1 + (\sin t)\vec{u}_2 \;=\; \boxed{(1,2,1) + \frac{\cos t}{\sqrt{5}}(1,2,0) + (\sin t)(0,0,1)}$$

5. The vector $(1,0,0)$ rotates to $\vec{u}_1 = \left(\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}\right)$, while the unit vector $\vec{u}_2 = (0,1,0)$ isn't affected by the rotation. Then the curve is

$$\vec{x}(t) \;=\; \left(1-\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right) + t\vec{u}_1 + t^2\vec{u}_2 \;=\; \left(1-\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right) + t\left(\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}\right) + t^2(0,0,1)$$

which can be simplified to $\boxed{(1,0,t^2) + \frac{t-1}{2}\big(\sqrt{3},0,-1\big)}$

Geometry of Space Curves

6. Note first that

$$\vec{x}\,'(t) \;=\; (2t+1, \cos t, e^t) \qquad \text{and} \qquad \vec{x}\,''(t) \;=\; (2, -\sin t, e^t).$$

so

$$\vec{x}\,'(0) \;=\; (1,1,1) \qquad \text{and} \qquad \vec{x}\,''(0) \;=\; (2,0,1).$$

Then $\vec{T}(0) \;=\; \dfrac{\vec{x}\,'(0)}{\|\vec{x}\,'(0)\|} \;=\; \boxed{\dfrac{1}{\sqrt{3}}(1,1,1)}$. Furthermore, the normal component of acceleration is

$$\vec{x}\,'' - (\vec{x}\,''\cdot\vec{T})\vec{T} \;=\; (2,0,1) - \frac{1}{3}\big((2,0,1)\cdot(1,1,1)\big)(1,1,1) \;=\; (1,-1,0)$$

so $\boxed{\vec{P}(0) = \dfrac{1}{\sqrt{2}}(1,-1,0)}$. Finally,

$$\vec{B}(0) \;=\; \vec{T}(0)\times\vec{P}(0) \;=\; \frac{1}{\sqrt{6}}\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} \;=\; \boxed{\frac{1}{\sqrt{6}}(1,1,-2)}$$

7. Recall that

$$\vec{x}\,'' \;=\; s''\,\vec{T} \,+\, (s')^2\kappa\,\vec{P}$$

We can get $\vec{T}$ by subtracting off the $\vec{P}$ component and then dividing by s'':

$$\vec{T} \;=\; \frac{\vec{x}\,'' - (\vec{x}\,''\cdot\vec{P})\vec{P}}{s''} \;=\; \frac{(5,2,-4) - \frac{1}{9}\big((5,2,-4)\cdot(2,2,-1)\big)(2,2,-1)}{-3} \;=\; \boxed{\frac{1}{3}(-1,2,2)}$$

8. Since the curve is unit-speed, $\vec{P}\,' = -\kappa\vec{T} + \tau\vec{B}$, so $\vec{P}\,'\times\vec{B} = \kappa\vec{P}$. Then

$$\vec{P} \;=\; \frac{\vec{P}\,'\times\vec{B}}{\|\vec{P}\,'\times\vec{B}\|} \;=\; \boxed{\frac{1}{\sqrt{34}}(-5,0,3)}$$

9. (a) The osculating circle has a radius of $1/\kappa = 6$, and is centered at $\vec{c} = \vec{p} + 6\vec{P} = (7,4,-3)$, so the parametrization is

$$\vec{c} + (6\cos t)\vec{T} + (6\sin t)\vec{P} \;=\; \boxed{(7,4,-3) + 3\sqrt{2}\cos t\,(1,0,1) + 2\sin t\,(2,1,-2)}$$

(b) The normal vector to the osculating plane is

$$\vec{B} \;=\; \vec{T}\times\vec{P} \;=\; \frac{1}{3\sqrt{2}}\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 0 & 1 \\ 2 & 1 & -2 \end{vmatrix} \;=\; \frac{1}{3\sqrt{2}}(-1,4,1)$$

The plane must go through the point $(3,2,1)$, so the equation is $\boxed{-x + 4y + z = 6}$

10. (**Note:** This problem originally had a typo. The correct value of $\vec{P}\,'$ is $(-1,-9,1)$.) The curve isn't unit-speed, so we must be careful using the Frenet-Serret formulas. We have that

$$\vec{P}\,' \;=\; s'\frac{dP}{ds} \;=\; s'(-\kappa\vec{T}+\tau\vec{B}) \;=\; -s'\kappa\,\vec{T} \,+\, s'\tau\,\vec{B}.$$

From $\vec{x}\,'(0)$, we know that $\vec{T}(0) = \dfrac{1}{\sqrt{2}}(0,1,-1)$ and $s'(0) = \sqrt{2}$, so

$$\kappa(0) \;=\; \frac{\vec{P}\,'\cdot\vec{T}}{-s'} \;=\; \frac{(-1,-9,1)\cdot\frac{1}{\sqrt{2}}(0,1,-1)}{-\sqrt{2}} \;=\; \boxed{5}$$

11. This one's tricky. Since the torsion is zero, the curve must be planar. Since the curvature is 1, the curve must be a unit circle. (You should definitely have gotten at least this far in the problem. It's ok if you didn't get the next part.) Since the acceleration of a unit circle is always the negative of the radial vector, the center is at $\vec{x}(0)+\vec{x}\,''(0) = (3,0,0)$, and it seems that

$$\vec{x}(t) \;=\; (3,0,0) + (\cos t)(-1,0,0) + (\sin t)(0,0,1).$$

Thus $\vec{x}(\pi/2) = \boxed{(3,0,1)}$

Parameterizations of Surfaces

12. (a) (**Note:** There was originally a typo in this part. The catenoid should be given by the equation $r = \cosh z$.)

$$\boxed{\vec{X}(u,v) = (\cosh u\cos v, \cosh u\sin v, u)}$$

(b) $\boxed{\vec{X}(u,v) = (u, 3\cos v, 3\sin v)}$

(c) $\boxed{\vec{X}(u,v) = (3\cos u\cos v, 2\sin u\cos v, 5\sin v)}$

(d) We have $r = 5+2\cos u$, $z = 1+2\sin u$, and $\theta = v$, so the parameterization is $\boxed{\vec{X}(u,v) = \big((5+2\cos u)\cos v, (5+2\cos u)\sin v, 1+2\sin u\big)}$

(e) $\boxed{\vec{X}(u,v) = (u, v, v^2+1)}$

(f) This is the surface $r^2 - z^2 = 4$. One method is to let $z = u$, $r = \sqrt{4+u^2}$, and $\theta = v$, giving

$$\boxed{\vec{X}(u,v) \;=\; \big(\sqrt{4+u^2}\,\cos v,\; \sqrt{4+u^2}\,\sin v,\; u\big)}$$

Another method is to use $r = 2\cosh u$, $z = 2\sinh u$, and $\theta = v$, which gives

$$\boxed{\vec{X}(u,v) \;=\; (2\cosh u\cos v,\; 2\cosh u\sin v,\; 2\sinh u)}$$

(g) One possibility is simply $\boxed{\vec{X}(u,v) = (u, u^2+v^2+2, v)}$. Another possibility is to use something like cylindrical coordinates, giving a parameterization such as $\boxed{\vec{X}(u,v) = (u\cos v, u^2+2, u\sin v)}$.

13. The unit vector $\vec{u}_1 = \dfrac{1}{\sqrt{3}}(1,1,1)$ goes in the direction of the line. To parameterize the cylinder, we need two more perpendicular unit vectors. We can just make up one of them, say $\vec{u}_2 = \dfrac{1}{\sqrt{2}}(1,-1,0)$, and taking the cross product gives the third vector:

$$\vec{u}_3 \;=\; \vec{u}_1 \times \vec{u}_2 \;=\; \frac{1}{\sqrt{6}}\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} \;=\; \frac{1}{\sqrt{6}}(1,1,-2)$$

(Alternatively, we probably could have guessed this vector.) Then the cylinder is

$$\begin{aligned} \vec{X}(u,v) \;&=\; u\,\vec{u}_1 \,+\, (2\cos v)\,\vec{u}_2 \,+\, (2\sin v)\,\vec{u}_3 \\ &=\; \boxed{\frac{u}{\sqrt{3}}(1,1,1) + \sqrt{2}\,\cos v\,(1,-1,0) + \frac{\sqrt{2}\,\sin v}{\sqrt{3}}(1,1,-2)} \end{aligned}$$

14. At time t, the rod is at $z = 3t$ and $\theta = 2t + \dfrac{\pi}{2}$. If we let $r = u$ and $t = v$, we get

$$\vec{X}(u,v) \;=\; \left(u\cos\left(2v+\frac{\pi}{2}\right), u\sin\left(2v+\frac{\pi}{2}\right), 3v\right) \;=\; \boxed{(-u\sin 2v, u\cos 2v, 3v)}$$

15. We have

$$\vec{X}_u(u,v) \;=\; \left(2u^{-1/2}\cos v, 2u^{-1/2}\sin v, 1\right) \quad \text{and} \quad \vec{X}_v(u,v) \;=\; \left(-\sqrt{u}\sin v, \sqrt{u}\cos v, 0\right).$$

The given point corresponds to $(u,v) = (2, 3\pi/4)$, so

$$\vec{X}_u \;=\; (-1/4,\; 1/4,\; 1) \qquad \text{and} \qquad \vec{X}_v \;=\; \left(-1,-1,0\right).$$

Taking the cross product gives a normal vector

$$\vec{X}_u \times \vec{X}_v \;=\; \frac{1}{4}\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -1 & 1 & 4 \\ -1 & -1 & 0 \end{vmatrix} \;=\; \boxed{(1,-1,1/2)}$$

Parametrizations in Higher Dimensions

16. There are several possibilities, such as

$$\boxed{\vec{X}(u,v,w) \;=\; (2+3\cos u\cos v\cos w,\; 3\sin u\cos v\cos w,\; 3\sin v\cos w,\; 3\sin w)}$$

or

$$\boxed{\vec{X}(u,v,w) \;=\; (2+3\cos u\cos w,\; 3\sin u\cos w,\; 3\cos v\sin w,\; 3\sin v\sin w)}$$

17. (a) The normal vector to the plane is $(1, 1, 2, 2)$. It's easy to make up two vectors perpendicular to this, e.g. $\vec{u}_1 = \boxed{\dfrac{1}{\sqrt{2}}(1, -1, 0, 0)}$ and $\vec{u}_2 = \boxed{\dfrac{1}{\sqrt{2}}(0, 0, 1, -1)}$. There are several different ways to find the third one, one of which is to use the version of cross product in $\mathbb{R}^4$:

$$\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \\ 1 & 1 & 2 & 2 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = (-4, -4, 2, 2)$$

so $\vec{u_3} = \boxed{\dfrac{1}{\sqrt{10}}(-2, -2, 1, 1)}$

(b) We use spherical coordinates with the three vectors:

$$\boxed{\begin{aligned} \vec{X}(u, v) &= (1, 2, 0, 0) + \frac{5\cos u \sin v}{\sqrt{2}}(1, -1, 0, 0) \\ &\qquad + \frac{5\sin u \sin v}{\sqrt{2}}(0, 0, 1, -1) + \frac{5\cos v}{\sqrt{10}}(-2, -2, 1, 1) \end{aligned}}$$

18. It looks like (x_1, x_2) can be any unit vector in $\mathbb{R}^2$, say $(\cos u, \sin u)$. Then (x_3, x_4) must be perpendicular to (x_1, x_2), which means it's a multiple of $(-\sin u, \cos u)$. Thus

$$\vec{X}(u, v) = \boxed{(\cos u, \sin u, -v \sin u, v \cos u)}$$

19. The unit normal to the hyperplane is $\vec{n} = \frac{1}{2}(1, -1, 1, -1)$. The point $(9, 0, 0, 0)$ lies on the hyperplane, so let $\vec{v} = (9, 0, 0, 0) - (1, 0, 0, 0) = (8, 0, 0, 0)$. Then the reflection is $(1, 0, 0, 0)$ plus twice the projection of $\vec{v}$ onto $\vec{n}$:

$$(1, 0, 0, 0) + 2(\vec{v} \cdot \vec{n})\vec{n} = (1, 0, 0, 0) + 2(2, -2, 2, -2) = \boxed{(5, -4, 4, -4)}$$

20. Let $u = {x_1}^2 + {x_2}^2 = {x_3}^2 + {x_4}^2$, let v be the angle of the point (x_1, x_2), and let w be the angle of the point (x_3, x_4). Then we get

$$\boxed{\vec{X}(u, v, w) = (u \cos v, u \sin v, u \cos w, u \sin w)}$$

Surface Integrals

21. We can parameterize this as $\vec{X}(u, v) = (u \cos v, u \sin v, u^2)$ with domain $0 < u < 1$ and $0 < v < 2\pi$. Then $\vec{X}_u$ and $\vec{X}_v$ will be perpendicular, so

$$\|\vec{X}_u \times \vec{X}_v\| = \|\vec{X}_u\| \, \|\vec{X}_v\| = \|(\cos v, \sin v, 2u)\| \, \|(-u \sin v, u \cos v, 0)\| = u\sqrt{1 + 4u^2}.$$

So the area is

$$\int_0^{2\pi}\int_0^1 u\sqrt{1+4u^2}\,du\,dv \;=\; 2\pi\left[\frac{1}{12}(1+4u^2)^{3/2}\right]_0^1 \;=\; \boxed{\frac{(5\sqrt{5}-1)\pi}{6}}$$

22. We can parameterize this as $\vec{X}(u,v) = (u\cos v, u\sin v, v)$ for $0 < u < 1$ and $0 < v < \pi$. Then $\vec{X}_u$ and $\vec{X}_v$ will be perpendicular, so

$$\|\vec{X}_u \times \vec{X}_v\| \;=\; \|\vec{X}_u\|\,\|\vec{X}_v\| \;=\; \|(\cos v, \sin v, 0)\|\,\|(-u\sin v, u\cos v, 1)\| \;=\; \sqrt{1+u^2}$$

Then the integral is

$$\int_0^{\pi}\int_0^1 y\|\vec{X}_u \times \vec{X}_v\|\,du\,dv \;=\; \int_0^{\pi}\int_0^1 (u\sin v)\sqrt{1+u^2}\,du\,dv$$
$$=\; \int_0^{\pi}\sin v\,dv\int_0^1 u\sqrt{1+u^2}\,du \;=\; 2\left[\frac{1}{3}(1+u^2)^{3/2}\right]_0^1 \;=\; \boxed{\frac{4\sqrt{2}-2}{3}}$$

23. We can parameterize this as $\vec{X}(u,v) = (u, v, e^u\sin v)$ for $0 < u < 1$ and $0 < v < \pi$. Then

$$\vec{X}_u \times \vec{X}_v \;=\; \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 0 & e^u\sin v \\ 0 & 1 & e^u\cos v \end{vmatrix} \;=\; (-e^u\sin v, -e^u\cos v, 1)$$

so $\|\vec{X}_u \times \vec{X}_v\| = \sqrt{1+e^{2u}}$. Then

$$\int_0^{\pi}\int_0^1 z^2\|\vec{X}_u \times \vec{X}_v\|\,du\,dv \;=\; \int_0^{\pi}\int_0^1 \left(e^{2u}\sin^2 v\right)\sqrt{1+e^{2u}}\,du\,dv$$
$$=\; \int_0^{\pi}\sin^2 v\,dv\int_0^1 e^{2u}\sqrt{1+e^{2u}}\,du \;=\; \frac{\pi}{2}\left[\frac{1}{3}(1+e^{2u})^{3/2}\right]_0^1 \;=\; \boxed{\frac{\pi\left((1+e^2)^{3/2}-2\sqrt{2}\right)}{6}}$$

24. We can parameterize this as $\vec{X}(u,v) = (u\cos v, u\sin v, u^2)$ with domain $0 < u < 2$ and $0 < v < 2\pi$. Then $\vec{X}_u$ and $\vec{X}_v$ will be perpendicular, so

$$\|\vec{X}_u \times \vec{X}_v\| \;=\; \|\vec{X}_u\|\,\|\vec{X}_v\| \;=\; \|(\cos v, \sin v, 2u)\|\,\|(-u\sin v, u\cos v, 0)\| \;=\; u\sqrt{1+4u^2}.$$

Then the integral is

$$\int_0^{\pi}\int_0^2 \sqrt{1+4z}\,\|\vec{X}_u \times \vec{X}_v\|\,du\,dv \;=\; \int_0^{\pi}\int_0^1 u(1+4u^2)\,du\,dv \;=\; \pi\left[\frac{u^2}{2}+u^4\right]_0^2 \;=\; \boxed{18\pi}$$

25. We can parameterize this as $\vec{X}(u,v) = (u, v, u^2)$ for $0 < u < 1$ and $0 < v < 3$. Then

$$\|\vec{X}_u \times \vec{X}_v\| \;=\; \|(1,0,2u)\times(0,1,0)\| \;=\; \sqrt{1+4u^2}$$

so the integral is

$$\int_0^3\int_0^1 xy\|\vec{X}_u\times\vec{X}_v\|\,du\,dv \;=\; \int_0^3\int_0^1 uv\sqrt{1+4u^2}\,du\,dv$$

$$= \int_0^3 v\,dv\int_0^1 u\sqrt{1+4u^2}\,du \;=\; \frac{9}{2}\left[\frac{1}{12}(1+4u^2)^{3/2}\right]_0^1 \;=\; \boxed{\frac{15\sqrt{5}-3}{8}}$$

26. This is the surface $r=e^z$, so we can parameterize this as $\vec{X}(u,v)=(e^u\cos v, e^u\sin v, u)$ for $0<u<1$ and $0<v<2\pi$. Then $\vec{X}_u$ and $\vec{X}_v$ will be perpendicular, so

$$\|\vec{X}_u\times\vec{X}_v\| \;=\; \|\vec{X}_u\|\,\|\vec{X}_v\| \;=\; \|(e^u\cos v, e^u\sin v, 1)\|\,\|(-e^u\sin v, e^u\cos v, 0)\| \;=\; e^u\sqrt{1+e^{2u}}.$$

Then the integral is

$$\int_0^{2\pi}\int_0^1 r\|\vec{X}_u\times\vec{X}_v\|\,du\,dv \;=\; \int_0^{2\pi}\int_0^1 e^{2u}\sqrt{1+e^{2u}}\,du\,dv$$

$$= 2\pi\left[\frac{1}{3}(1+e^{2u})^{3/2}\right]_0^1 \;=\; \boxed{\frac{2\pi\big((1+e^2)^{3/2}-2\sqrt{2}\big)}{3}}$$

Types of Parameterizations

27. (a) We have $\vec{X}_u=(-\sin u, 0, \cos u)$ and $\vec{X}_v=(0,1,0)$, which are perpendicular unit vectors, so this parameterization is $\boxed{\text{both conformal and equiareal, i.e. an isometry}}$.

(b) We have $\vec{X}_u=(\sinh u\cos v, \sinh u\sin v, 1)$ and $\vec{X}_v=(-\cosh u\sin v, \cosh u\cos v, 0)$. These vectors are perpendicular, with $\|\vec{X}_u\|=\|\vec{X}_v|=\cosh u$, so this parameterization is $\boxed{\text{conformal}}$.

(c) We have $\vec{X}_u=(1,0,1)$ and $\vec{X}_v=(0,2/3,1/3)$. These aren't perpendicular, but

$$\|\vec{X}_u\times\vec{X}_v\| \;=\; \|(-2/3,-1/3,2/3)\| \;=\; 1$$

so this parameterization is $\boxed{\text{equiareal}}$.

(d) We have $\vec{X}_u=(\cos v,\sin v, 2u)$ and $(-u\sin v, u\cos v, 0)$. These vectors are perpendicular, but their lengths are $\|\vec{X}_u\|=\sqrt{1+4u^2}$ and $\|\vec{X}_v\|=u$, so this parameterization is $\boxed{\text{neither conformal nor equiareal}}$.

(e) We have $\vec{X}_u=(10u, 6v, 8v)$ and $\vec{X}_v=(-10v, 6u, 8u)$. These vectors are perpendicular, with $\|\vec{X}_u\|=\|\vec{X}\|_v=10\sqrt{u^2+v^2}$, so this parameterization is $\boxed{\text{conformal}}$.

28. We have $\vec{X}_u=\big(ke^{ku}\cos v, ke^{ku}\sin v, ke^{ku}\big)$ and $\vec{X}_v=\big(-e^{ku}\sin v, e^{ku}\cos v, 0\big)$. These vectors are always perpendicular, with

$$\|\vec{X}_u\|=k\sqrt{2}\,e^{ku} \qquad\text{and}\qquad \|\vec{X}_v\|=e^{ku}$$

which will have the same length when $\boxed{k=1/\sqrt{2}}$.

29. We have $\vec{X}_u = (-2k \sin ku, 2k \cos ku, 0)$ and $\vec{X}_v = (0, 0, 3)$. These are always perpendicular, with

$$\|\vec{X}_u \times \vec{X}_v\| \;=\; \|\vec{X}_u\|\,\|\vec{X}_v\| \;=\; (2k)(3) = 6k.$$

Thus, this will be equiareal when $\boxed{k = 1/6}$.