Exam 2 Practice Problems

Math 352, Fall 2014

Parameterizing Space Curves

1. The unit circle in the xy-plane is rotated $90°$ around the line $y = x$. Find parametric equations for the resulting space curve.

2. The catenary $y = \cosh x$ in the xy-plane is reflected across the plane $y = 3z$. Find parametric equations for the resulting curve.

3. Find parametric equations for the ellipse whose minor axis has endpoints $(2, 4, 4)$ and $(-2, -4, -4)$ and whose minor axis has endpoints $(2, 1, -2)$ and $(-2, -1, 2)$.

4. Find parametric equations for the unit circle centered at $(1, 2, 1)$ that lies in the plane $y = 2x$.

5. The parabola $y = x^2$ in the xy-plane is rotated $30°$ around the line $x = 1$, with the vertex of the parabola moving to the point $\left(1-\dfrac{\sqrt{3}}{2}, 0, \dfrac{1}{2}\right)$. Find parametric equations for the resulting space curve.

Geometry of Space Curves

6. Let $\vec{x}\colon \mathbb{R} \to \mathbb{R}^3$ be the curve $\vec{x}(t) = (t^2 + t, \sin t, e^t)$. Compute the Frenet vectors $\vec{T}$, $\vec{P}$, and $\vec{B}$ at the point $(0, 0, 1)$.

7. On a space curve $\vec{x}(t)$, suppose that

$$\vec{x}\,''(0) = (5, 2, -4), \qquad s''(0) = -3, \qquad \text{and} \qquad \vec{P}(0) = \frac{1}{3}(2, 2, -1).$$

   Compute $\vec{T}(0)$.

8. On a unit-speed space curve $\vec{x}(t)$, suppose that $\vec{P}\,'(0) = (3, 4, 5)$ and $\vec{B}(0) = (0, 1, 0)$. Compute $\vec{P}(0)$.

9. Let $\vec{x}(t)$ be a space curve that goes through the point $\vec{p} = (3, 2, 1)$, and suppose that the tangent vector at $\vec{p}$ is $\frac{1}{\sqrt{2}}(1, 0, 1)$, the principle normal vector at $\vec{p}$ is $\frac{1}{3}(2, 1, -2)$, and the curvature at $\vec{p}$ is $1/6$.

   (a) Find parametric equations for the osculating circle at the point $\vec{P}$.

   (b) Find a Cartesian equation for the osculating plane at the point $\vec{P}$.

10. On a space curve $\vec{x}(t)$, suppose that $\vec{x}\,'(0) = (0, 1, -1)$ and $\vec{P}\,'(0) = (-1, -9, 1)$. Compute $\kappa(0)$.

11. A unit-speed curve $\vec{x}\colon \mathbb{R} \to \mathbb{R}^3$ has constant curvature 1 and constant torsion 0. Given that $\vec{x}(0) = (2,0,0)$, $\vec{x}\,'(0) = (0,0,1)$, and $\vec{x}\,''(0) = (1,0,0)$, compute $\vec{x}(\pi/2)$.

Parameterizations of Surfaces

12. Find parameterizations for the following surfaces:

    (a) The catenoid $r = \cosh z$.

    (b) The cylinder centered along the x-axis with a radius of 3.

    (c) The ellipsoid $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{5}\right)^2 = 1$.

    (d) The torus $(r-5)^2 + (z-1)^2 = 4$.

    (e) The parabolic cylinder $z = y^2 + 1$.

    (f) The hyperboloid $x^2 + y^2 - z^2 = 4$.

    (g) The paraboloid $y = x^2 + z^2 + 2$.

13. Find parametric equations for the cylinder of radius 2 whose axis is the line $x = y = z$.

14. A long rod initially lies along the y-axis. The rod begins rotating counterclockwise (around the z-axis) at a rate of 2 rad/sec while simultaneously moving in the z-direction at a rate of 3 units/sec. Find parametric equations for the surface traced out by the rod.

15. Let S be the surface parameterized by $\vec{X}(u,v) = (\sqrt{u}\cos v, \sqrt{u}\sin v, u)$. Find a normal vector to S at the point $(-1,1,2)$.

Parametrizations in Higher Dimensions

16. Find a parametrization of the 3-sphere in $\mathbb{R}^4$ defined by the equation

$$(x_1-2)^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = 9.$$

17. Let H be the hyperplane $x_1 + x_2 + 2x_3 + 2x_4 = 3$ in $\mathbb{R}^4$.

    (a) Find three orthogonal unit vectors that are parallel to H.

    (b) Find a parameterization $\vec{X}(u,v)$ for the sphere of radius 5 in H centered at the point $(1,2,0,0)$.

18. Let S be the surface in $\mathbb{R}^4$ defined by the equations

$${x_1}^2 + {x_2}^2 = 1 \qquad \text{and} \qquad x_1x_3 + x_2x_4 = 0.$$

    Find a parametrization of S.

19. Find the reflection of the point $(1,0,0,0)$ across the hyperplane $x_1 - x_2 + x_3 - x_4 = 9$.

20. Find a parametrization of the 3-manifold in $\mathbb{R}^4$ defined by the equation

$$x_1{}^2 + x_2{}^2 \;=\; x_3{}^2 + x_4{}^2.$$

Surface Integrals

21. Let S be the portion of the paraboloid $z = x^2 + y^2$ lying below the plane $z = 1$. Find the surface area of S.

22. Compute $\displaystyle\iint_S y\,dA$, where S is the portion or the helicoid $z = \theta$ satisfying $0 < r < 1$ and $0 < \theta < \pi$.

23. Compute $\displaystyle\iint_S z^2\,dA$, where S is the portion of the surface $z = e^x \sin y$ satisfying $0 < x < 1$ and $0 < y < \pi$.

24. Evaluate $\displaystyle\iint_S \sqrt{1+4z}\,dA$, where S be the portion of the paraboloid $z = x^2 + y^2$ lying below the plane $z = 4$.

25. Evaluate $\displaystyle\iint_S xy\,dA$, where S is the portion of the surface $z = x^2$ for which $0 < x < 1$ and $0 < y < 3$.

26. Let S be the surface $z = \ln(r)$ in the range $0 < z < 1$. Evaluate $\displaystyle\iint_S r\,dA$.

Types of Parameterizations

27. Determine whether each of the following parameterizations is equiareal, conformal, both, or neither.

    (a) $\vec{X}(u,v) \;=\; (\cos u,\; v,\; \sin u)$
    (b) $\vec{X}(u,v) \;=\; (\cosh u \cos v,\; \cosh u \sin v,\; u)$
    (c) $\vec{X}(u,v) \;=\; \left(u,\; \tfrac{2}{3}v,\; u + \tfrac{1}{3}v\right)$
    (d) $\vec{X}(u,v) \;=\; (u\cos v,\; u\sin v,\; u^2)$
    (e) $\vec{X}(u,v) \;=\; (5u^2 - 5v^2, 6uv, 8uv)$

28. Find a constant k so that

$$\vec{X}(u,v) = \left(e^{ku}\cos v, e^{ku}\sin v, e^{ku}\right)$$

is a conformal parametrization of the cone $z = \sqrt{x^2+y^2}$.

29. Find a constant k so that

$$\vec{X}(u,v) = (2\cos ku, 2\sin ku, 3v)$$

is an equiareal parametrization of the cylinder $x^2 + y^2 = 4$.