Math 352 Name: SOLUTION

Exam 2

1. [**12 points**] Let C be the space curve $\vec{x}(t) = (\cos 2t, \sin 3t, \sin 4t)$. Compute the curvature of C at the point $(1, 0, 0)$.

$$\vec{x}\,'(t) = (-2\sin 2t, 3\cos 3t, 4\cos 4t)$$

$$\vec{x}\,''(t) = (-4\cos 2t, -9\sin 3t, -16\sin 4t)$$

$$\vec{x}\,'(0) = (0, 3, 4) \qquad \vec{T}(0) = \tfrac{1}{5}(0, 3, 4) \qquad s'(0) = 5$$

$$\vec{x}\,''(0) = (-4, 0, 0)$$

$$\vec{x}\,'' = s''\,\vec{T} + \kappa (s')^2 \vec{P}$$

But $\vec{x}\,''(0)$ is orthogonal to $\vec{T}(0)$.

So $\kappa (s')^2 \vec{P} = (-4, 0, 0)$

$$\kappa (s')^2 = 4$$

$$\kappa = \frac{4}{5^2} = \boxed{\frac{4}{25}}$$

2. **[14 points]** Evaluate $\iint_S z\,dA$, where S is the portion of the cone $z = \sqrt{x^2+y^2}$ in the range $0 < z < 3$.

$\vec{X}(u,v) = (u\cos v,\ u\sin v,\ u)$ $\quad 0 < u < 3$, $\quad 0 < v < 2\pi$

$\vec{X}_u = (\cos v,\ \sin v,\ 1)$

$\vec{X}_v = (-u\sin v,\ u\cos v,\ 0)$

$\|\vec{X}_u \times \vec{X}_v\| = \|\vec{X}_u\|\,\|\vec{X}_v\| = \sqrt{2}\,u$

$$\iint_S z\,dA = \int_0^{2\pi}\int_0^3 u(\sqrt{2}u)\,du\,dv$$

$$= 2\pi \int_0^3 u^2\sqrt{2}\,du$$

$$= 2\pi(9\sqrt{2}) = \boxed{18\pi\sqrt{2}}$$

3. [**12 points**] Find a constant k so that

$$\vec{X}(u,v) = \left(k\sqrt{u}\cos v,\ k\sqrt{u}\sin v,\ k\sqrt{u}\right)$$

is an equiareal parametrization of the cone $z = \sqrt{x^2+y^2}$.

$$\vec{X}_u = \left(\frac{k}{2\sqrt{u}}\cos v,\ \frac{k}{2\sqrt{u}}\sin v,\ \frac{k}{2\sqrt{u}}\right)$$

$$\vec{X}_v = \left(-k\sqrt{u}\sin v,\ k\sqrt{u}\cos v,\ 0\right)$$

$$\|\vec{X}_u \times \vec{X}_v\| = \|\vec{X}_u\|\,\|\vec{X}_v\|$$

$$= \left(\frac{k}{2\sqrt{u}}\sqrt{2}\right)\left(k\sqrt{u}\right) = \frac{k^2}{\sqrt{2}}$$

equiareal: $\dfrac{k^2}{\sqrt{2}} = 1$

$$k^2 = \sqrt{2}$$

$$\boxed{k = \sqrt[4]{2}} \quad (\text{or } k = -\sqrt[4]{2})$$

4. [**14 points**] The paraboloid $z = x^2 + y^2$ is rotated slightly so that its axis is the line $x = y = z$, with the vertex of the paraboloid staying fixed at $(0, 0, 0)$. Find parametric equations for the resulting surface.

$$(0,0,1) \to \frac{1}{\sqrt{3}}(1,1,1)$$

$$(1,0,0) \to \frac{1}{\sqrt{2}}(1,-1,0) \quad \text{(or any vector} \perp \text{to } (1,1,1)\text{)}$$

$$(0,1,0) \to \frac{1}{\sqrt{6}}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \frac{1}{\sqrt{6}}(1,1,-2)$$

$$\boxed{\vec{X}(u,v) = \frac{u}{\sqrt{2}}(1,-1,0) + \frac{v}{\sqrt{6}}(1,1,-2) + \frac{u^2+v^2}{\sqrt{3}}(1,1,1)}$$

5. **[18 points]** On a unit-speed space curve, $\vec{T}'(0) = (2, 0, 2)$ and $\vec{T}''(0) = (9, 7, 1)$. What is $\kappa'(0)$?

$$\vec{T}' = \kappa \vec{P} \qquad (2,0,2) = \kappa \vec{P}$$

so $\kappa = 2\sqrt{2}$, $\vec{P} = \frac{1}{\sqrt{2}}(1,0,1)$

$$\vec{T}'' = \kappa' \vec{P} + \kappa \vec{P}'$$

$$\kappa' = \vec{T}'' \cdot \vec{P}$$

$$= (9,7,1) \cdot \frac{1}{\sqrt{2}}(1,0,1)$$

$$= \boxed{5\sqrt{2}}$$

6. [**15 points**] Let P be the plane in $\mathbb{R}^4$ parameterized by

$$\vec{X}(u,v) \;=\; (u+5v,\; u-v,\; u+5v,\; u+7v).$$

Find a parametrization for the unit circle on P centered at the origin.

$$\vec{X}_u = (1,1,1,1)$$
$$\vec{X}_v = (5,-1,5,7)$$

These are parallel to the plane, but not orthogonal to each other

$$\vec{u}_1 = \frac{1}{2}(1,1,1,1)$$

$$\vec{X}_v - (\vec{X}_v \cdot \vec{u}_1)\vec{u}_1 = (5,-1,5,7) - \frac{8}{2}(1,1,1,1)$$
$$= (1,-5,1,3)$$

$$\vec{u}_2 = \frac{1}{6}(1,-5,1,3)$$

$$\boxed{\vec{x}(t) = \frac{\cos t}{2}(1,1,1,1) + \frac{\sin t}{6}(1,-5,1,3)}$$

7. **[15 points]** Let $\vec{X}\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the function

$$\vec{X}(u,v) \;=\; \left(uv,\, u^2+9v^2,\, u^2+3v^3\right)$$

Find all critical points of $\vec{X}$.

$$d\vec{X} = \begin{bmatrix} v & u \\ 2u & 18v \\ 2u & 9v^2 \end{bmatrix} \qquad \begin{bmatrix} u \\ 18v \\ 9v^2 \end{bmatrix} = c \begin{bmatrix} v \\ 2u \\ 2u \end{bmatrix}$$

$$u = Cv$$
$$18v = 2Cu$$
$$9v^2 = 2Cu$$

$$9v^2 = 18v$$
$$\Rightarrow v=0 \text{ or } v=2$$

$\underline{v=0}$

$$\begin{bmatrix} 0 & u \\ 2u & 0 \\ 2u & 0 \end{bmatrix} \qquad u=0$$

$\underline{v=2}$

$$\begin{bmatrix} 2 & u \\ 2u & 36 \\ 2u & 36 \end{bmatrix} \qquad \begin{bmatrix} u \\ 36 \\ 36 \end{bmatrix} = c \begin{bmatrix} 2 \\ 2u \\ 2u \end{bmatrix} \qquad \begin{matrix} u = 2C \\ 36 = 2Cu \end{matrix} \Rightarrow 36 = u^2$$
$$\Rightarrow u = \pm 6$$

$$u=6: \begin{bmatrix} 2 & 6 \\ 12 & 36 \\ 12 & 36 \end{bmatrix} \checkmark \qquad u=-6: \begin{bmatrix} 2 & -6 \\ -12 & 36 \\ -12 & 36 \end{bmatrix} \checkmark$$

$$\boxed{(0,0),\ (-6,2),\ \text{and}\ (6,2)}$$